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Blow-ups of Two-Convex, Type-I Mean Curvature Flow

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I am submitting herewith a dissertation written by Kevin Michael Sonnanburg entitled "Blow-ups of Two-Convex, Type-I Mean Curvature Flow." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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Blow-ups of Two-Convex, Type-I Mean Curvature Flow

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Abstract

Much of geometric analysis can be described as the study of (hyper)surfaces changing shape subject to certain equations. Here we study one such equation, mean curvature flow, which decreases the area of a surface as fast as possible. However, solutions to this equation develop singularities. I present a detailed analysis of this development under suitable restrictions on curvature.

Assuming mean-convexity and type-I growth of curvature in time, there are three main parts to the results:

1) I collect well-known results to describe the shape of (rescalings) blow-ups near singularities in a specific way with high precision.

2) Colding and Minicozzi showed uniqueness of blow-ups and Andrews showed a restriction on the surface collapsing. I combine these to ensure the formation of a certain neck shape to emulate the neck-pinching argument of Angenent to control when the singularity occurs. In turn I use this to show the conditions at the singular time depend in a nice way on the initial conditions.

3) Since blow-ups are often stationary, their study leads to the study of ancient solutions, which exist for all negative time. I classify all possible ancient solutions arising from blow-ups under certain conditions.

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Chapter 1

Introduction

1.1 Setup

1.1.1 Geometric Analysis

Geometric analysis is the study of how geometric differential equations affect geometric objects. A parabolic differential equation, such as the heat equation, is diffusive and typically has a regularizing effect. Thus if one is to evolve an irregular hypersurface by a parabolic geometric partial differential equation, one expects the hypersurface to become smoother and more regular.

This work looks at one such evolution equation, mean curvature flow. Imagine a rather irregular soap bubble floating in air. The bubble tries to reduce its surface area rapidly, often forcing it to become more round. A (hyper)surface moving according to mean curvature flow is similar to the motion of the bubble, without air resistance. A bubble similar in shape to a sphere will become rounder and shrink to a single point.

Before diving into the specifics of the equation for mean curvature flow, see Figure 1.1 for an illustration of the idea. Consider a curve in $\mathbb{R}^2$. Mean curvature for a curve is just the curvature. Each point on the curve moves with velocity proportional to the curvature vector at that point (in this case mean curvature flow is called curve-shortening flow). (See Figure

1.1) In other words, move each point in the direction of the curvature vector (which may be inward or outward), with speed equal to the magnitude of curvature, then the overall flow can be fast-forwarded or slowed.

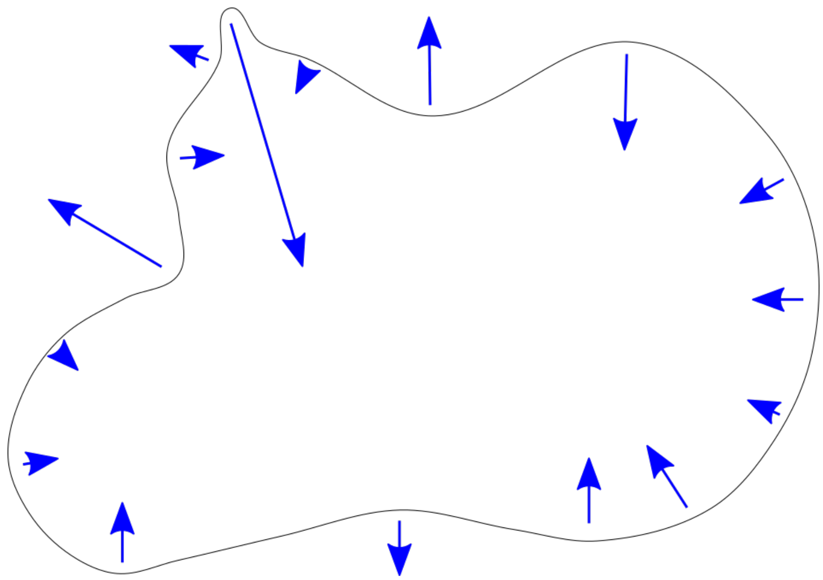

Figure 1.1: Initial curve moving by mean curvature flow.

Over time, the curve continues to change according to the mean curvature flow equation. The curve eventually tends inward and disappears, as in Figure 1.2.

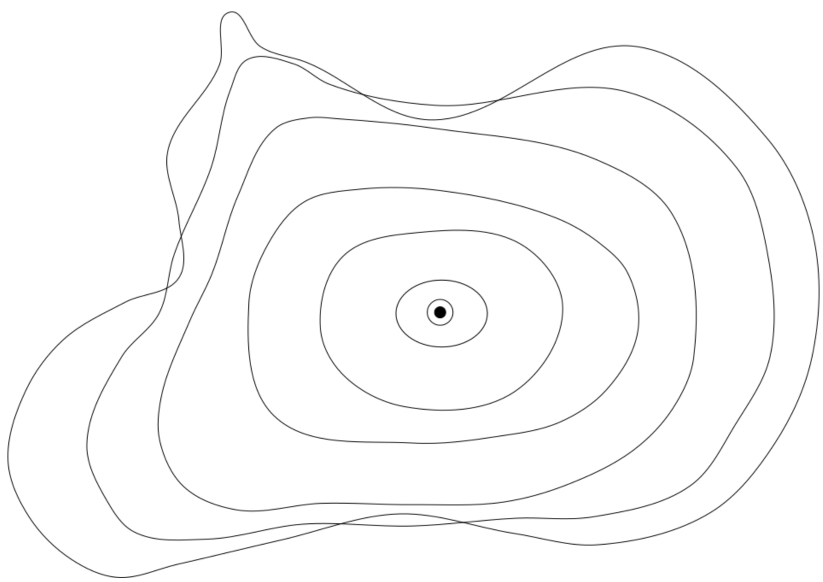

Figure 1.2: Curve moving by curve-shortening flow at multiple times.

Mean curvature flow has seen applications in relativity, image analysis, minimal surfaces and cell biology. Its study probably began with the investigation of the evolution of grain boundaries by Mullins in [39].

Another way to understand the motion of mean curvature flow is as the gradient flow for the area functional. That is, the area of $M(t)$ is decreasing as fast as possible, locally and therefore globally.

1.1.2 Mean Curvature

Let $\mathcal{M}$ be an smooth, closed abstract N-dimensional manifold and consider

$$\mathbf{F} : \mathcal{M} \times [0, T) \to \mathbb{R}^{N+1},$$

a family of smooth embeddings of $\mathcal{M}$ into $\mathbb{R}^{N+1}$, indexed by $t \in [0, T)$. Write $M(t)$ for each hypersurface that is the image of $\mathbf{F}(\mathcal{M}, t)$, and define the flow $M = \{M(t)\}_{t\in[0,T)}$.

For $(p, t) \in \mathcal{M} \times [0, T)$ the mean curvature vector at each point $x = \mathbf{F}(p, t) \in M(t)$ is

$$\mathbf{H}(x, t) = (\Delta_{M(t)}\mathbf{F})(p, t),$$

where $\Delta_{M(t)}$ is the Laplace-Beltrami operator on $\mathcal{M}(t)$. The Laplace-Beltrami operator can be thought of as the Laplace operator intrinsic to $M(t)$ so that it is invariant to orientation and parameterization. In fact, let $\Sigma \subset \mathbb{R}^{N+1}$ be a surface that can be expressed as a graph over a hyperplane P of a smooth function $f : P \to \mathbb{R}$, with ν a choice of normal vector to P. Then if $\phi : P \to \Sigma$ is defined by $\phi(x) = x + f(x)\nu$, we have $\Delta_\Sigma \phi = \Delta f$ at each critical point of f. That is, the Laplace-Beltrami operator is essentially the Laplace operator but adjusted for orientation and parameterization-invariance.

The mean curvature vector $\mathbf{H}$ is always normal to the hypersurface. Since $M(t)$ is embedded, we can make a choice of normal vector $\nu(x, t)$ that points outward everywhere.

Then if we write H for the scalar mean curvature so that

$$\mathbf{H} = -H\nu,$$

a sphere will have positive mean curvature, that is $H > 0$ at every point on the sphere.

Thus the mean curvature H can be thought of as the average of curvatures in all directions. In fact, if $g(t)$ is the metric on $\mathcal{M}$ induced by $\mathbf{F}(\cdot, t)$, then H is the trace of the second fundamental form A, that is $H = \operatorname{tr}_g A$. For our purposes, the second fundamental form can be thought of as a matrix describing all curvatures. In fact,

$$A_{ij}(x,t) = -\partial_i \partial_j \mathbf{F}(x,t) \cdot \nu(x,t).$$

1.1.3 Mean Curvature Flow

Now define a motion on $M(t)$ so that each point moves with velocity equal to the mean curvature vector:

$$\partial_t \mathbf{F}(p,t) = \mathbf{H} = -H(x,t)\nu(x,t).$$

Under this flow, a sphere will tend inward uniformly and vanish in finite time (this follows from a direct ODE calculation), and a plane will be stationary. A saddle point can be close to stationary for an interval of time while the rest of the hypersurface is more mobile.

Mean Curvature Flow is shown to be well-posed in several settings, but in this work we are concerned with closeness of initial data in the sense that one initial hypersurface is a graph over the other. Well-posedness in this setting is shown in the sense of graphs in the first chapter of [37], which follows §7 of [26]. Mean curvature flow is parabolic, so the diffusive behavior is not prone to chaos. There are some stability results for the sphere, by Escher and Simonett in [15] or particular neck shapes by Gang, Knopf, and Sigal in [21, 20, 19]. However, little is known about the long time behavior based on initial conditions. That is the motivation for this work.

It is known that compact flows must arrive at singularities in finite time. As with other parabolic equations, there is a comparison theorem that keeps hypersurfaces from colliding (See Proposition 2.4 of [14]). Thus any compact hypersurface can be placed inside a larger sphere that will contract to a point in finite time. Since the hypersurface inside the sphere must avoid the sphere, it must become singular no later than the sphere does. Thus for each compact embedded flow M there is a first singular time $T > 0$ at which the curvature becomes unbounded.

Later we will distinguish between what are called type-I and type-II singularities. Type-I flows exhibit a natural bound on curvature growth

$$|A| \leq C(2(T-t))^{-\frac{1}{2}},$$

with $C > 0$. This is the rate at which the curvature of a sphere blows up and can be found from solving the ODE gotten by assuming uniform curvature. Uniform curvature generally indicates a uniform roundness, such as a sphere or generalized cylinder (that is $\mathbb{S}^k \times \mathbb{R}^{N-k}$, though we will just say cylinder). One might then expect blow-ups of type-I singularities to be approximately round. Due to previous results discussed in the next section, this expectation can be backed up for mean-convex (that is, $H > 0$) hypersurfaces.

Type-II singularities are less well understood, and there are few known examples. They are also less well-behaved. However my hope is that "most" mean curvature flows are type-I, rendering the assumption of the type-I bound as a hypothesis quite reasonable. "Most" here means some sense of openness and density of the set of initial data leading to type-I singularities.

1.2 Singularities

We consider a singularity to be where curvature becomes unbounded. Due to Proposition 2.4.9 of [37], $\mathbf{F}$ cannot become singular without $|A|$ blowing up, so that notion of singularity is plenty sufficient. The matter of singularities in geometric analysis is more delicate than a typical PDE since the domain of curvature is a moving target. Therefore by singular point we mean a point $x \in \mathbb{R}^{N+1}$ that is the limit of a sequence $(x_i, t_i) \in M(t_i) \times [0, T)$ such that $|A(x_i, t_i)| \to \infty$. Alternatively, one can consider a sequence pair (p_i, t_i) so that curvature at $\mathbf{F}(p_t, t_i)$ blows up. Incidentally, in the type-I, mean-convex case we consider, Stone showed in Theorem 3.1 of [41] that if $p_i \to p \in \mathcal{M}$, then $\mathbf{F}(p, t_i)$ will also develop a singularity. This simplifies much of the discussion around singular points, since it mitigates some of the moving target aspect.

1.2.1 Possible singularities

The first major result regarding singularities of mean curvature flow was observed in Huisken's seminal paper [27]. There he showed any initial hypersurface that is uniformly convex (that is, the eigenvalues of the second fundamental form A are strictly positive) must not only shrink to a single point, but that its blow-up tends asymptotically to a round sphere. A separate argument for curves was needed and given by Gage and Hamilton in [18], though it was also shown by Grayson in [23] that any embedded curve must eventually become convex. Thus the essentials of singularities for curves are understood completely.

On the other hand, if $N > 1$, it is not guaranteed that the hypersurface will contract to a single point. Angenent demonstrated the existence of neck-like singularities in [6] by showing the existence of a homothetically contracting torus that he uses to pinch the neck of prescribed initial data, as in Figure 1.3.

In fact, without the assumption that $H \geq 0$, the existence of mean curvature flows that shrink homothetically (which are therefore type-I) of higher genus was shown by Chopp in [10], and a particularly strange example can be found in [34]. Thus mean-convexity is also a reasonable assumption if one wants any semblance of control on the behavior of singularities.

Less is known about type-II singularities, since one has generally less control on velocity and curvature. There however are a number of examples of type-II flows. Notable earlier examples can be found by Angenent and Velázquez in [5] and Hamilton's dumbbell given by Angenent, Altschuler, and Giga in [1].

1.2.2 Flowing Past Singularities

This work only deals with the smooth case, but there do exist multiple successful definitions of weak flows to continue the flow past a singularity. Chen, Giga, and Goto in [9] and Evans and Spruck in [16] independently developed a theory for level set flows, wherein $M(t)$ is considered a level set of an evolving function $\phi : \mathbb{R}^{N+1} \to \mathbb{R}$.

A much weaker formulation of Brakke found in [7] is measure-theoretic. One can consider an N-dimensional Radon measure on $\mathbb{R}^{N+1}(t)$ concentrated entirely on $M(t)$. The scope of the Brakke flow is wide, but the disadvantage is a lack of unique solutions.

Huisken and Sinestrari showed in [30] the efficacy of surgeries at singularities. They develop a complex theory of neck formation demonstrating the preservation of varying

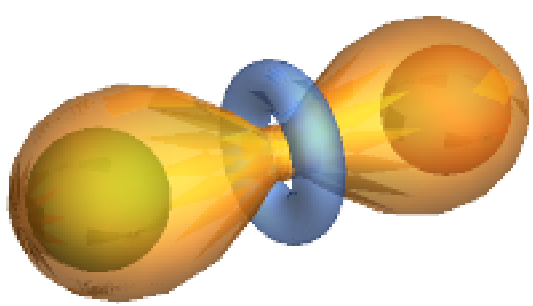

Figure 1.3: Balls large enough to outlive the donut prop open the "bulbs" as the donut pinches the neck.

degrees of convexity and roundness. A near-singular neck can be removed and replaced with two round caps on the two remaining connected components of the hypersurface.

1.2.3 Type-I Singularities

As mentioned, the type-I bound on curvature is a natural bound to assume that we expect to result in round blow-ups. This assumption has two clear advantages.

The first is a control of the velocity. Integrating velocity and employing the type-I, one finds that $\mathbf{F}(p,t)$ converges to a limit point as $t \to T$ like $\sqrt{T-t}$ goes to 0. The second is control of all orders of curvature. In [28] Huisken used induction on a maximum principle to show that under an appropriate rescaling (discussed in the next subsection), all orders of curvature are uniformly bounded in time, affording uniform convergence of any order of blow-ups on compact sets. He follows an argument of Langer in [35] who proved a compactness theorem for a certain class of bounded surfaces.

1.2.4 Blow-ups

There are two common types of blow-ups we will consider. The first we call the *rescaled flow*, which is merely a natural rescaling of a mean curvature flow according to the type-I rate. The second is a sequence of flows that are rescaled by successively larger scales and converge to another flow; we call the limit flow a *tangent flow*.

In the rescaled flow, a singular point is recentered to the origin and the whole hypersurface is dilated at a rate inverse to that of a sphere contracting (time is rescaled to an interval $[s_0, \infty)$ and the rescaled flow is no longer a mean curvature flow). Thus we hope a rescaled flow of a type-I mean curvature flow will converge to something round. As mentioned before, that will be the case in the mean-convex case, as shown by Huisken. In fact, due to the curvature bounds from the type-I assumption, the convergence is smooth on compact subsets. However, the limits in time shown to exist by Huisken in [28] are only subsequential.

For the round case we are interested in, Colding and Minicozzi showed in [12] the uniqueness we want, but for tangent flows. In this setting, time and space are rescaled by a sequence of constants around a fixed point in space-time, corresponding to a sequence of flows that are all rescaled but also solutions to the mean curvature flow equation. If the flows converge to a flow, then that flow is a tangent flow. However, in this case convergence is in the weak sense of measures. White showed in [43] that in the mean-convex case that tangent flows must be cylinders, spheres, or hyperplanes. Sheng and Wang rule out the hyperplane case in [40] when going forward to the first singular time.

The rescaled flows are simpler to work with and we will see they are like a special case of tangent flows. Furthermore, the convergence we get when using the rescaled flow is much stronger than the weak measure-theoretic convergence, so we mostly use the rescaled flow. We only need tangent flows to use the uniqueness result of Colding and Minicozzi, so we briefly discuss the connection to the rescaled flow and then stick to the rescaled flow. Since Huisken in [28, 29] showed the limits of the rescaled flow are round in the type-I, mean-convex case, there is only the worry that the orientation of the limit cylinder may depend on subsequence. The uniqueness result assuages precisely that issue.

1.3 Questions

At this point, there are certainly questions that are natural to ask about these blow-ups, some of which have been touched on, but let's collect them here. Initial short-time questions:

- Do these blow-up limits even exist?
- What might they be?
- Are they unique?
- Do they represent some sense of equilibrium?
- How nice are they: smooth, embedded, regular?

Then there are long-time questions to ask:

- Is there a monotone energy that is useful?
- When and how do singularities occur?
- How does the singular time depend on initial conditions?
- How do conditions at the singular time depend on initial conditions?
- Are the equilibria locally stable?
- Are there neighborhoods of initial data with consistent behavior (such as type-I-ness)?
- Are there more precise asymptotics for the shape near singularities?
- Are any of these phenomena globally stable in any sense?

Using curvature bounds for type-I mean curvature flows, Huisken showed that subsequential limits of the rescaled flow exist and exhibit smooth convergence on compact subsets of $\mathbb{R}^{N+1}$. Ilmanen showed in [32] that all sequences used for tangent flows do converge subsequentially to some tangent flow in the weak measure-theoretic sense. For type-I, mean-convex mean curvature flows Huisken showed in [28] that the subsequential limits of the rescaled flow centered at a singular point are nonempty and either hyperplanes, cylinders, or spheres. Again, between White and Sheng and Wang, we know the limit is a cylinder or sphere.

Note that White does not need the type-I assumption for classifying the geometries of singularities for compact, mean-convex flows, but the convergence is weaker. Moreover, we don't strictly need the roundness result for tangent flows, since we have it for the rescaled flow, but it makes the argument in Chapter 4 simpler. As mentioned before, we *do* need tangent flows for the uniqueness result of Colding and Minicozzi.

Huisken found in [28] that the area function with an ambient Gaussian weight can be used as a monotone energy for mean curvature flow. Since the energy is decreasing and bounded below by 0, he observed that the derivative of the Gaussian area must converge to 0. It happens that the integrand in that derivative is the normal motion of the flow. Thus setting it equal to 0 corresponds to an equilibrium for the rescaled flow, or a homothetically shrinking mean curvature flow. As mentioned above, Huisken then showed that the only mean-convex hypersurfaces (subject to some niceness conditions) are the round ones, and that is how he determines the roundness of blow-ups. Since the convergence in this case is smooth, the blow-ups inherit the smoothness and regularity of the rescaled flow. In addition, Andrews showed in [2] that balls with radius inversely proportional to the curvature can be placed tangent to the hypersurface, both inside and out, prevent certain types of collapse and also self-intersection in the limit. Intuitively, it prevents a hypersurface from locally collapsing down on its osculating sphere and also prevents global collapse by keeping relatively flat parts of the hypersurface away from each other.

This dissertation answers questions about continuity of blow-ups with respect to initial conditions, but very little is otherwise known about the long-time behavior of mean curvature flow. There are, however, a handful of results regarding the stability of certain blow-ups, in particular of spheres and cylinders. In fact, Colding and Minicozzi show in [11] that spheres and hyperplanes are stable with respect to a certain "entropy" related to the aforementioned energy. Escher and Simonett showed the local stability of the sphere (convergence here is slightly weaker so this does not follow from Huisken's convex flow result since there is no assumption of convexity). Gang, Knopf, and Sigal showed the stability of a certain profile around the cylinder with precise asymptotics. However, these only work in certain neighborhoods of these equilibria. Since the blow-ups we deal with are all either cylinders or spheres, my eventual plan is to demonstrate some sense of global stability to the profile near the cylinder, following an argument similar to that of Kammerer, Merle, and Zaag in [33]. In the process, the next step after this dissertation is to show that type-I, mean-convex initial conditions are open, perhaps even dense.

In this dissertation, I synthesize the aforementioned results regarding roundness of blow-ups to ensure the formation of a specific neck structure for type-I, mean-convex, two-dimensional mean curvature flows (unless the singularity is spherical). Then I use a geometric argument to show that the two bulbs outside the neck to do not become entirely singular, and thus there is space in them. I am then able to emulate Angenent's neck-pinching argument, with a sphere in each bulb, to show the first singular time is continuous with respect to initial data. From there it follows easily that the limit set $M(T)$ is Hausdorff-continuous with respect to initial data.

Neck formations have already been heavily investigated. There is the aforementioned collapsing donut strategy of Angenent in [6] to show the development of neck-pinches for surfaces. The two-dimensional restriction is required since in higher dimensions it is possible for a singularity to look like a generalized cylinder with more than one flat factors, allowing too much freedom in the flat direction. Later Huisken and Sinestrari developed a theory around neck formation showing properties of different degrees of convexity regarding the lowest eigenvalues of A (the lowest principal curvatures). In particular, a two-convex hypersurface (the sum of the lowest two eigenvalues is positive) remains two-convex and two-convexity is required for a true neck-like structure (close to $\mathbb{S}^{N-1} \times \mathbb{R}^1$). Earlier, Angenent, Altschuler, and Giga studied rotationally symmetric solutions to mean curvature flow in [1]. In that case they were able to show the precise profile of the limit hypersurface $M(T)$ at a singular point. Furthermore all those singularities are type-I!

1.4 Ancient Solutions

An ancient solution to a geometric PDE is one that exists for all time before some time t_0. Since translation in time has no effect, we will just say ancient means it exists for all negative time. Notice that a type-I bound for all negative time actually forces curvature to decay as $t \searrow -\infty$ (which still means bounded curvature for the rescaled flow). Investigation of ancient solutions is already well under way. Huisken and Sinestrari showed in [31] that ancient, type-I (for all negative time), compact solutions must be spheres. Angenent found in [3] a non-trivial ancient solution to curve-shortening flow by using matched asymptotics to glue together two copies of the translating solution called the "grim reaper". This was generalized to higher dimensions by Haslhofer and Hershkovitz in [24] and further by Angenent, Daskalopoulos, and Šešum with asymptotics in [4]. What's more, Daskalopoulos, Huisken, and Šešum classified all embedded, compact, ancient, convex solutions for curve-shortening flow. My goal is a similar result for higher dimensions without the compactness assumption. Currently my argument only works for two dimensions because of differing values of the natural energy at certain equilibria. It is a kind of Liouville theorem inspired by that of Giga and Kohn in [22] for a modified heat equation, although my technique follows more closely that of Huisken showing limits for forward time in [28].

Why are ancient solutions relevant? Since we expect our blow-ups to give rise to equilibria, such equilibria should be ancient solutions. Indeed, since at any time an ancient solution has had infinite time for diffusion to act, they should be highly regular. Hopefully their behavior can give insight to restrictions on blow-ups. In fact, there is a way, as Merle and Zaag use in [38] to rescale a sequence of solutions so that the limit is ancient, and so must inherit the nice properties mentioned. This technique allows for assuming the curvature conditions I need for the Liouville theorem. Thus, it is important to my planned strategy to show openness of type-I initial data.

Finally, since I follow Huisken's subsequential limit argument, I need control on higher order curvature going back in time. It is standard for parabolic equations to find that regularity at a point in space-time provides regularity on a parabolic space-time ball preceding the point in time, and we use the analogous result from [14]. I make a special argument for ancient solutions using the type-I bound to extend the parabolic ball to arbitrary negative time.

Chapter 2

The Essentials

2.1 Notation

Throughout this paper, we recycle the use of the following letters.

- $\mathcal{M}$: background manifold
- p: point in $\mathcal{M}$
- x, y: point in $\mathbb{R}^{N+1}$
- M: family of hypersurfaces flowing by mean curvature
  (i.e. $M = \{M(t)\}_{t\in[0,T)}$)
- $\mathbf{F}$: parameterization $\mathbf{F} : \mathcal{M} \times [0,T) \to M$
- $\mathbf{F}^{-1}$: preimage of $x \in \mathbb{R}^{N+1}$ under $\mathbf{F}(\cdot, t)$.
  That is, for given t, $\mathbf{F}^{-1}(\cdot, t) : M(t) \to \mathcal{M}$, and $p = \mathbf{F}^{-1}(x, t)$.
- Σ: fixed hypersurface
- $\Omega(t)$: open region enclosed by $M(t)$
  (definable for closed, embedded hypersurfaces)
- ν: unit *outward* normal vector

- T: first singular time
- A: second fundamental form
- H: mean curvature
- d_H: Hausdorff distance

The following diacritics are applied to any of the above to associate them to a specific hypersurface.

- $\Box_0$: initial data
- $\bar{\Box}$: associated with the "target" flow $\overline{M}$ (i.e. $M_{0n} \to \overline{M}_0$)
  (we use $Cl(\cdot)$ for closure, not a bar)
- $\Box_n$: associated with M_n
- $\widetilde{\Box}$: associated with the rescaled flow $\widetilde{M}$ (as in §2.2)
- $\widehat{\Box}$: associated with some auxiliary flow $\widehat{M}$ or hypersurface $\widehat{\Sigma}$ local to a proof
- $\Box_{\asymp}$: associated with the neck of a flow
  (See Definition 5.7)
- $\Box_{=\!\circ/\circ\!=}$: associated with a bulb of the flow
  (See Definition 5.11)
- $p^* := \lim_{t \to T} \mathbf{F}(p, t) \in \mathbb{R}^{N+1}$
  (exists due to Lemma 3.2)
- M^*: limit set consisting of all such points p^*
- $\Box^*$: associated with M^*

Remark 2.1. *The definition of M^* given here is equivalent to that in Lemma 3.3 by Lemma 3.4.*

The following are used in the setting of the rescaled flow.

- $\lambda(t) = (2(T-t))^{-\frac{1}{2}}$ is the scaling factor
- $\xi = \lambda x$ is the new spatial variable
- $s = -\frac{1}{2}\log(T-t)$ is the rescaled time
- $s_0 = -\frac{1}{2}\log T$ is the initial time for $\widetilde{M}$

We also use $\cong$ for homeomorphicity and $\subset$ for *strict* set inclusion.

2.2 Definitions

Graph Convergence For *smooth, closed hypersurfaces*, we say $\Sigma_n \to \Sigma$ as a graph to order k if the following holds (see Figure 2.1):

Let Σ and Σ_n be smooth, closed hypersurfaces. Assume there is a C^∞ function $f_n : \Sigma \to \mathbb{R}$ so that the map $\varphi_n(x) = x + f_n(x)\nu(x)$ is a smooth diffeomorphism from Σ to Σ_n. Then, for every $k \in \mathbb{N}$, $\|f_n\|_{C^k} \xrightarrow[n\to\infty]{} 0$.

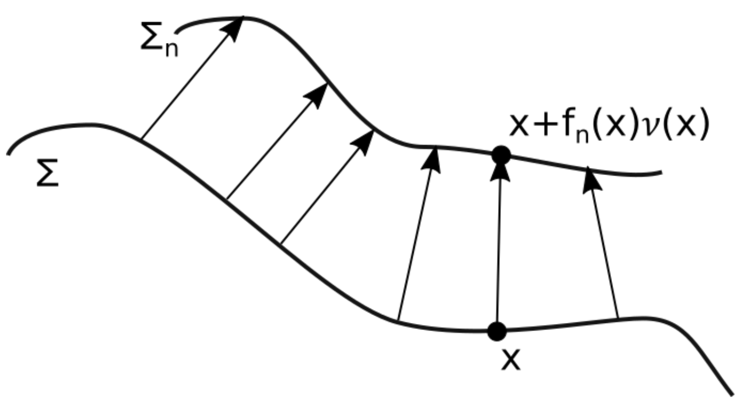

Figure 2.1: Σ_n is close to Σ.

Remark 2.2. *Since $M_{n0} \cong \overline{M}_0 \cong \mathcal{M}$, we can consider $\mathbf{F}_n$ to have $\mathcal{M}$ as its background manifold as well. This works up to time T_n, since the flow preserves embeddings (see §2.3).*

If d_H is the Hausdorff distance, then $\|f\|_{C^0} \geq d_H$ (see Lemma 3.1).

Remark 2.3. *We would make frequent use of Huisken's Theorem 3.4 in [28], which is a compactness result for hypersurfaces whose conclusion gives convergence in the manner above. However, the theorem only tells us the limit hypersurface is immersed. With Lemma 4.12, we ensure our blow-up limits are embedded using the* ***Non-Collapsing Condition*** *in §2.3.*

Singular Point We say $x \in \mathbb{R}^{N+1}$ is a singular point if there is a sequence $(p_i, t_i) \in \mathcal{M} \times [0, T)$ such that $\mathbf{F}(p_i, t_i) \to x$ and $|A(p_i, t_i)| \to \infty$ as $i \to \infty$.

All mentions of singularities are at the first singular time T.

Limit Set Let $M(t) \subset \mathbb{R}^{N+1}$ be a compact mean curvature flow defined for times $t \in [0, T)$, where T is the first singular time. Define the *limit set* M^* to be the set of points $x \in \mathbb{R}^{N+1}$ such that there exists a sequence of times $t_i \nearrow T$ and a sequence of points $x_i \in M(t_i)$, where $x_i \to x$.

Properly Embedded Let $\mathcal{M}$ be an N-dimensional manifold and $\mathbf{F} : \mathcal{M} \to \mathbb{R}^{N+1}$ be a parameterization of a hypersurface $\Sigma \subset \mathbb{R}^{N+1}$. We say Σ is *properly embedded* if, for every compact subset K of $\mathbb{R}^{N+1}$, $\mathbf{F}^{-1}(K \cap \Sigma)$ is compact in $\mathcal{M}$.

Polynomial Volume Growth We say a surface $\Sigma \in \mathbb{R}^3$ has *polynomial volume growth* if $Vol(B_R(0) \cap \Sigma)$ is bounded by some polynomial $P(R)$. (Here Vol is the volume *intrinsic* to Σ, or the "hypersurface area".)

We say a mean curvature flow M has *uniform polynomial volume growth* if, for every t that M is defined, $M(t)$ has polynomial volume growth, where the polynomial $P(R)$ is independent of t.

Type-I Singularities We say M is a type-I flow if for some $C > 0$,

$$\max_{M(t)} |A(p,t)| \leq C \left(2(T-t)\right)^{-\frac{1}{2}} \text{ for } t \in [0, T),$$

Of course, since $|H| \leq \sqrt{N}|A|$, we can say the same of H, for a different C.

Remark 2.4. *The type-I condition is typically employed in discussions of blow-ups at singularities. However we apply the condition to the entirety of an ancient flow, meaning curvature decays as $t \searrow -\infty$ as well.*

Cylinders Since much of this work is set specifically in $\mathbb{R}^3$, it is convenient to distinguish between cylinders and their generalizations.

By *generalized cylinder* we mean any set $(\sqrt{m}\,\mathbb{S}^m) \times \mathbb{R}^{N-m}$ with $1 \leq m \leq N-1$ (up to isometry).

By *cylinder*, we mean specifically $\mathbb{S}^1 \times \mathbb{R}$, (up to isometry).

(Note, the radii are fixed.)

Rescaled Flow To understand the asymptotic behavior near the singularity, we consider the rescaled flow:
If $x \in \mathbb{R}^{N+1}$ is a singular point of M,

$$\begin{aligned} &\widetilde{\mathbf{F}}_x(p,s) := \lambda(t)\left(\mathbf{F}(p,t) - x\right), \\ &\text{where } \lambda(t) = (2(T-t))^{-\frac{1}{2}} \text{ and } s = -\frac{1}{2}\log(T-t), \end{aligned} \tag{2.1}$$

for $s \in [s_0, \infty)$, where $s_0 = -\frac{1}{2}\log(T)$ and we use $\xi = \lambda x$ as the spatial variable. We will refer to this rescaling as "the rescaled flow".

As introduced in [28], $\widetilde{\mathbf{F}}_x$ solves

$$\partial_s \widetilde{\mathbf{F}}_x = \widetilde{\mathbf{F}}_x - \widetilde{H}_x \widetilde{\nu}_x, \tag{2.2}$$

where $\widetilde{H}_x = \widetilde{H}\left(\widetilde{\mathbf{F}}_x(p,s)\right)$ and $\widetilde{\nu}_x = \widetilde{\nu}\left(\widetilde{\mathbf{F}}_x(p,s)\right)$. Objects associated with the rescaled flow are indicated by a tilde. For simplicity, we will mostly be dealing with the flow rescaled around the origin, meaning $x = 0$. In that case we will *omit* the subscript: $\widetilde{\mathbf{F}} := \widetilde{\mathbf{F}}_0$.

Remark 2.5. *If M is type-I, then $\widetilde{H}$ uniformly bounded for all time, even if M is ancient.*

Tangent Flow Let M be a mean curvature flow for times $t \in [0, T)$. Fix $(x_0, t_0) \in \mathbb{R}^{N+1} \times [0, T)$. Then for $\mu > 0$, one can check that the rescaling

$$M_{\mu,(x_0,t_0)}(t) := \mu\left(M\left(t_0 - \mu^{-2}(-t)\right) - x_0\right) \text{ for } t \in \left[-\mu^2 t_0, \mu^2(T - t_0)\right)$$

is also a solution to mean curvature flow. Taking a sequence $\mu_i \nearrow \infty$, consider the sequence of rescalings $M^i_{(x_0,t_0)}(t) := M_{\mu_i,(x_0,t_0)}(t)$. If $M^i_{(x_0,t_0)}(t)$ has a subsequence converging in i to a flow $M^\infty(t)$, that limit is called a *tangent flow*. The convergence is for varifolds in the weak measure-theoretic sense (see Appendix C). Ilmanen showed in [32] that tangent flows always exist for $\mu_i \to \infty$ for mean curvature flow. In Appendix C we show that the smooth convergence in the graph sense above is much stronger than that convergence for measures, and we will only need that sense smooth convergence for this work. From Remark 2.6 below, we see that we can use the rescaled flow in place of tangent flows so we do not detail varifold convergence here. However, we define tangent flows here because theorems we use require tangent flows in their hypotheses.

It will be most convenient to consider tangent flows at $(x_0, t_0) = (0, T)$, so we typically *omit* the subscript:

$$M^i(t) = M^i_{(0,T)}(t) = \mu_i M(T + \mu_i^{-2} t) \text{ for } t \in [-\mu_i^2 T, 0). \tag{2.3}$$

Remark 2.6. *For our tangent flow, let $(x_0, t_0) = (0, T)$. Now choose some sequence $t_i \nearrow T$, with corresponding $s_i \nearrow \infty$. Recalling $\lambda(t) = (2(T - t))^{-\frac{1}{2}}$, let $\mu_i = \lambda(t_i)$. Then $\mu_i \nearrow \infty$. Then, using the notation from the* rescaled flow *and* tangent flow *definitions from §2.2, we make the observation that, by (2.3)*

$$\begin{aligned} M^i\left(-\frac{1}{2}\right) &= \mu_i M\left(T - \mu_i^{-2}\left(\frac{1}{2}\right)\right) \\ &= \lambda(t_i) M\left(T - \frac{1}{2}\left(2(T - t_i)\right)\right) = \lambda(t_i) M\left(T - \frac{1}{2}\lambda^{-2}(t_i)\right) = \lambda(t_i) M(t_i) = \widetilde{M}(s_i), \end{aligned} \tag{2.4}$$

with the rescaled flow on the right, and the tangent flow rescalings on the left.

Gaussian Area For a flow $M(t)$ of surfaces in $\mathbb{R}^3$, define

$$E_{(x_0,t_0)}(t) = \int_{M(t)} \rho_{(x_0,t_0)}(x,t)\, d\mu,$$

where

$$\rho_{(x_0,t_0)}(x,t) = \frac{1}{(4\pi(t_0-t))^{\frac{N}{2}}} e^{\frac{-|x-x_0|^2}{4(t_0-t)}}.$$

We will mostly be assuming $(x_0,t_0) = (0,0)$. In that case, we *omit* the subscript. That is, $E := E_{(0,0)}$ and $\rho := \rho_{(0,0)}$.

2.3 Tools

The following are previously established results, and will be taken for granted throughout this work.

Well-posedness (Theorem 1.5.1 of [37])

Given M_0 is compact and immersed (we require it to be embedded anyway), well-posedness for mean curvature flow has been established in multiple contexts, but the classical case is nicely laid out in §1.5 of [37], along with the PDE background in Appendix A of the same work.

Theorem 2.7 (Well-posedness)**.** *For any initial, smooth, compact hypersurface in $\mathbb{R}^{N+1}$ given by an immersion $\mathbf{F}_0 : \mathcal{M} \to \mathbb{R}^{N+1}$, there exists a unique, smooth solution to mean curvature flow $\mathbf{F} : \mathcal{M} \times [0,T)$ for some $T > 0$, with $M_0 = \mathbf{F}_0(\mathcal{M})$.*

Moreover, the solution depends smoothly on the initial immersion $\mathbf{F}_0$.

Convergence in the last sentence is in the graph sense. That is, if M_{n0} is the graph of a smooth function f_{n0} over $\overline{M}_0$, then there is a $\delta > 0$ such that for each $t \in [0,\delta)$, $M_n(t)$ is a graph over $\overline{M}(t)$. Furthermore, for any nonnegative integer k, if $\|f_{n0}\|_{C^k(\overline{M}_0)} \to 0$, then

$\|f_n(t)\|_{C^k(\overline{M}(t))} \to 0$ for each fixed $t \in [0, \delta)$. Since conditions at each time can be thought of as new initial data, this process can be repeated for larger and larger n, so that we have the same convergence at later nonsingular times and

$$\liminf_{n\to\infty} T_n \geq \overline{T}.$$

Therefore, in showing continuity of first singular time, we need only show that $\limsup_{n\to\infty} T_n \leq \overline{T}$.

In particular, $k = 2$ gets us uniform convergence of $H_n(t)$ to $\overline{H}(t)$. Precisely, this means that if $M_n(t)$ is a graph of f_n over $\overline{M}(t)$, so that for $x \in \overline{M}(t)$ there is $y_n = x + f_n(x)\nu$, then $H_n(y_n) \to \overline{H}(x)$. The convergence is uniform since $\overline{M}(t)$ is compact. Due to Lemma 3.1, $k = 0$ also gets us convergence of $M_n(t)$ to $\overline{M}(t)$ in the Hausdorff distance.

Embedding Preservation (Theorem 2.2.7 of [37])

If the initial hypersurface is compact and embedded, then it remains embedded during the flow.

In particular, for any $t_1, t_2 \in [0, T)$, $M(t_1) \cong \mathcal{M} \cong M(t_2)$. For example, a simply connected surface would stay simply connected for the duration of the flow.

Minimum Principle (Proposition 2.4.1 of [37])

The evolution equation for H is $\partial_t H = \Delta H + |A|^2 H$.

Mean-convexity is preserved by mean curvature flow, since $H \geq 0$ implies that $\partial_t H \geq \Delta H$. In fact, H immediately becomes positive everywhere, and $\min_{M(t)} H$ strictly increasing in t, when $H_0 \geq 0$. This means that H is strictly positive for $t \in (0, T)$, and bounded away from 0 after any short time.

Comparison Principle (Corollary 2.2.3 of [37])

Similar to comparison principles for other parabolic equations, initially disjoint solutions remain disjoint. More specifically, let M_1 and M_2 be compact, embedded mean curvature flows, with respective first singular times T_1 and T_2. Assume M_2 begins strictly inside M_1.

Then that containment is preserved until time $\min\{T_1, T_2\}$.

Due to the subsequent discussion in [37] (right before Corollary 2.2.5), this can be extended to allow the hypersurfaces to touch initially. Even if the initial hypersurfaces coincide in some places, but some of M_2 is strictly inside M_1, they will immediately be disjoint after any short amount of time. Then Corollary 2.2.3 of [37] can again be applied.

Non-Collapsing Condition (Theorem 3 of [2])

From Definition 1 of [2]: We say a mean-convex hypersurface M_0 bounding an open region Ω in $\mathbb{R}^{N+1}$ is α-non-collapsed if, for every $x \in M_0$, there exists a sphere of radius $\frac{\alpha}{H(x)}$ contained in $Cl(\Omega)$, and another contained in Ω^c, tangent to M_0 at x. (See Figure 2.2). If a hypersurface M_0 is α-non-collapsed, we have that the condition is preserved, with the same α, by mean curvature flow up to the first singular time, and in this case we also say the flow M is α-non-collapsed. Moreover, the non-collapsing relation $r = \dfrac{\alpha}{H}$ is scale-invariant, so if a mean curvature flow M is α-non-collapsed, so is $\widetilde{M}$.

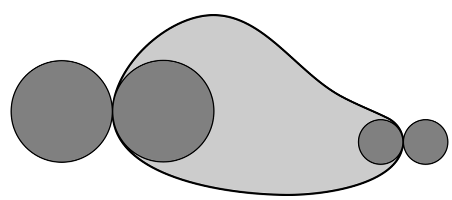

Figure 2.2: Spheres of radii varying with curvature.

Hopf Link The following scenario takes place in $\mathbb{R}^3$ at a fixed time $t_0 \in [t_{\asymp}, T)$, so we *omit* time in the present discussion of the Hopf link. We assume M has a cylindrical singularity (see Definition 4.20) at the origin, with the x_2-axis as the axis of the cylinder. (For more details on the notation, see Lemma 5.5 and Definition 5.7.)

More than once, we use the notion of the Hopf link to "trap" part of a curve or surface in a pinching neck to force a singularity, as in Figure 2.3a. Assume M is not simply connected. We will have isolated a "neck" of M as the intersection between it and a truncated, filled cylinder K. Within K is $\mathbb{D} = K \cap \{x_2 = 0\}$. The boundary of $\mathbb{D}$ is a circle, which is a simple, closed curve. Now take another simple, closed curve γ lying in M that passes through $\mathbb{D}$, transversely, exactly once.

The two curves γ and $\mathbb{D}$ form a Hopf link, which is a nontrivial link in $\mathbb{R}^{N+1}$. To see this, rotate coordinates so the page is the x_1x_3-plane and the x_2-axis has its positive direction coming out of the page, and $\mathbb{D}$ lies in the page, as depicted in Figure 2.3b. Consider only $\mathbb{D}$ and γ, but extend K to the infinite cylinder $K' = \{\sqrt{x^2+y^2} < 4\lambda\}$, so that for a point in K', $x_2 > 0$ and $x_2 < 0$ correspond to being "in front of" and "behind" $\mathbb{D}$, respectively (see Figure 2.3b). Note the inequality is strict so that K' is open, unlike K. That way we need not count the linking numbers when γ merely touches the boundary of K'.

Let us compute the linking number, considering γ to be going clockwise, (that is, the projection of γ into the x_1x_3-plane has a positive winding number with respect to the origin). As illustrated, if γ leaves K' while $x_2 > 0$ or enters K' while $x_2 < 0$, we add 1. Contrastly, we subtract 1 any time γ enters K' while $x_2 > 0$ or leaves K' while $x_2 < 0$. Since γ intersects $\mathbb{D}$ at only one point, x_2 only changes sign once while γ is in K'. This means we need only count the first time γ leaves K', and the last time it enters, since all other times at which it enters, it must exit on the same side of the x_1x_3-plane, and they cancel. The sum is thus two, and the linking number is one.

By Lemma 5.5, $M \cap \partial\mathbb{D} = \varnothing$, so $\gamma \cap \partial\mathbb{D} = \varnothing$. Thus neither curve can contract to a point via homotopy, if the embedding is to be preserved. The curve γ undergoes homotopy because the flow is continuous. The curve $\partial\mathbb{D}$ shrinks homothetically, by its construction in Definition 5.7, which is also a homotopy. Therefore, the link is preserved up to time T.

(a) Two (thick) curves making a Hopf link

(b) Sum: 2, Linking number: 1

Figure 2.3: Hopf link

Chapter 3

Preliminaries

3.1 Some Technical Lemmas

The business of convergence and singularities in geometric analysis is very delicate without any control on a flow. In this subsection we use the type-I and mean-convex assumptions to rein in bad behavior and make subsequent discussion more intuitive. The expert reader may wish to skip the proofs.

Lemma 3.1. *Let Σ and $\widehat{\Sigma}$ be hypersurfaces such that $\widehat{\Sigma}$ is a graph of a smooth function f over Σ so that the map $\varphi : x \mapsto x + f(x)\nu$ is a diffeomorphism from Σ to $\widehat{\Sigma}$. Then $d_H(\widehat{\Sigma}, \Sigma) \leq \|f\|_{C^0(\Sigma)}$.*

Proof. For $x \in \Sigma$, one can see that

$$|f(x)| = |x - \varphi(x)| \geq \inf_{y \in \widehat{\Sigma}} |x - y|.$$

Then

$$\|f\|_{C^0(\Sigma)} = \sup_{x \in \Sigma} |f(x)| \geq \sup_{x \in \Sigma} \inf_{y \in \widehat{\Sigma}} |x - y|.$$

Now let $g(y) := f\left(\varphi^{-1}(y)\right)$ for $y \in \widehat{\Sigma}$, so that

$$|g(y)| = |\varphi^{-1}(y) - y| \geq \inf_{x \in \Sigma} |x - y|.$$

Since φ is a diffeomorphism, that means

$$\|f\|_{C^0} = \sup_{x\in\Sigma} |f(x)| = \sup_{y\in\widehat{\Sigma}} |g(y)| \geq \sup_{y\in\widehat{\Sigma}} \inf_{x\in\Sigma} |x-y|.$$

Thus we have

$$\|f\|_{C^0} \geq \max\left\{\sup_{x\in\Sigma} \inf_{y\in\widehat{\Sigma}} |x-y|, \sup_{y\in\widehat{\Sigma}} \inf_{x\in\Sigma} |y-x|\right\} = d_H(\Sigma, \widehat{\Sigma}).$$

□

Next we draw from the proof of Lemma 3.3 of [28] to gain more control on $\mathbf{F}(p,t)$ for a given p.

Lemma 3.2. *Let M be a type-I mean curvature flow with first singular time T. Let $C > 0$ be the type-I constant, i.e. $|A| \leq C\lambda$.*

Then for each $p \in \mathcal{M}$ there is a $p^* \in \mathbb{R}^3$ for which

$$|\mathbf{F}(p,t) - p^*| \leq C\left(2(T-t)\right)^{\frac{1}{2}} \leq C\lambda^{-1}(t),$$

and therefore

$$|\widetilde{\mathbf{F}}_{p^*}(p,s)| \leq C$$

for all $s \in [s_0, \infty)$.

Proof. Choose some $0 < t_0 \leq t_1 \leq t_2 < T$. Given the type-I bound $H \leq C\left(2(T-t)\right)^{-\frac{1}{2}}$ for some $C > 0$,

$$\begin{aligned}\left|\int_{t_1}^{t_2} H\left(\mathbf{F}(p,\tau)\right)\, d\tau\right| &\leq \int_{t_1}^{t_2} C\left(2(T-\tau)\right)^{-\frac{1}{2}}\, d\tau \\ &\leq C\left|\left(2(T-t_1)\right)^{\frac{1}{2}} - \left(2(T-t_2)\right)^{\frac{1}{2}}\right| \leq 2C\left(2(T-t_0)\right)^{\frac{1}{2}}.\end{aligned}$$

Then for every sequence $t_i \nearrow T$, $\mathbf{F}(p, t_i)$ is a Cauchy sequence. Thus there is a $p^* := \lim_{t \to T} \mathbf{F}(p, t) \in \mathbb{R}^{N+1}$ exists. Now,

$$|p^* - \mathbf{F}(p,t)| = \left| \int_t^T \partial_\tau \mathbf{F}(p, \tau)\, d\tau \right| \leq C\lambda^{-1}(t).$$

and we are done. $\square$

Lemma 3.2 is used in Corollary 3.6 to show the rescaled flow $\widetilde{M}(s)$ does not drift off to spatial infinity. Otherwise the limit $\widetilde{M}_{-\infty}$ found in Theorem 6.12 would instead be empty!

Lemma 3.3 (Proposition 2.2.6 of [37]). *The limit set M^* is compact. Furthermore, $x \in M^*$ if and only if for every $t \in [0, T)$, the closed ball of radius $\sqrt{2N(T-t)}$ and center x intersects $M(t)$.*

This lemma will be useful more than once in §2.

Lemma 3.4. *Let M be a type-I mean curvature flow with first singular time T. Let $C > 0$ be the type-I constant. Then the set M^*, as in 17 2.2, is the same as $\{p^*\}_{p \in \mathcal{M}}$.*

Proof. For every $p \in \mathcal{M}$, $p^* := \lim_{t \to T} \mathbf{F}(p, t)$, so $p^* \in M^*$ follows by definiton.

Now let $x \in M^*$. Then there is a sequence $(x_i, t_i) \in \mathbb{R}^{N+1} \times [0, T)$ for which $t_i \nearrow T$ and $x_i \to x$. Since $\mathbf{F}$ is a homeomorphism, there must be $p_i \in \mathcal{M}$ such that $x_i = \mathbf{F}(p_i, t_i)$. Since $\mathcal{M}$ is compact, there is a subsequence indexing i (which we don't relabel) such that $p_i \to p$ in $\mathcal{M}$.

Let $\varepsilon > 0$. Choose $t_0 \in [0, T)$ so that $2C\left(2(T - t_0)\right)^{\frac{1}{2}} < \varepsilon$. By continuity of $\mathbf{F}$, there is $\delta > 0$ such that $|\mathbf{F}(p, t_0) - \mathbf{F}(q, t_0)| < \varepsilon$ whenever $\mathrm{dist}_{\mathcal{M}}(p, q) < \delta$. Now there is i_0 such that whenever $i \geq i_0$, we have $|x - x_i| < \varepsilon$, $\mathrm{dist}_{\mathcal{M}}(p, p_i) < \delta$, $t_i \geq t_0$, and $|\mathbf{F}(p, t_i) - p^*| < \varepsilon$, with the last inequality by Lemma 3.2. Assume $i \geq i_0$ so that we know

$$|x - x_i| < \varepsilon, \; |\mathbf{F}(p_i, t_0) - \mathbf{F}(p, t_0)| < \varepsilon, \text{ and } |\mathbf{F}(p, t_i) - p^*| < \varepsilon.$$

By the proof of Lemma 3.2, we also have that

$$|\mathbf{F}(p_i,t_i)-\mathbf{F}(p_i,t_0)|<\varepsilon \text{ and } |\mathbf{F}(p,t_0)-\mathbf{F}(p,t_i)<\varepsilon.$$

Recalling that $x_i=\mathbf{F}(p_i,t_i)$

$$\begin{aligned}
|x-p^*| &= |x-x_i+\mathbf{F}(p_i,t_i)-\mathbf{F}(p_i,t_0)+\mathbf{F}(p_i,t_0)-\mathbf{F}(p,t_0)\\
&\qquad +\mathbf{F}(p,t_0)-\mathbf{F}(p,t_i)+\mathbf{F}(p,t_i)-p^*|\\
&\le |x-x_i|+|\mathbf{F}(p_i,t_i)-\mathbf{F}(p_i,t_0)|+|\mathbf{F}(p_i,t_0)-\mathbf{F}(p,t_0)|\\
&\qquad +|\mathbf{F}(p,t_0)-\mathbf{F}(p,t_i)|+|\mathbf{F}(p,t_i)-p^*|\\
&< 5\varepsilon.
\end{aligned}$$

Since ε was arbitrary, we must have $|x-p^*|=0$, and we are done. □

3.2 Some Calculus

Lemma 3.5. *If M is a compact, type-I mean curvature flow, then $\widetilde{M}(s)$ is uniformly bounded for all times $s\le s_0$.*

Proof. From the type-I bound, we know $|\widetilde{H}|\le C_0$. Going back in time,

$$\begin{aligned}
\partial_{-s}|\widetilde{\mathbf{F}}|^2 &= -\partial_s|\widetilde{\mathbf{F}}|^2 = -2\widetilde{\mathbf{F}}\cdot\partial_s\widetilde{\mathbf{F}} = -2\widetilde{\mathbf{F}}\cdot(\widetilde{\mathbf{F}}-\widetilde{H}\widetilde{\nu})\\
&= -2\widetilde{\mathbf{F}}\cdot\widetilde{\mathbf{F}}+2\widetilde{\mathbf{F}}\cdot\widetilde{H}\widetilde{\nu}\le -2|\widetilde{\mathbf{F}}|^2+2C_0|\widetilde{\mathbf{F}}|.
\end{aligned}$$

So $\partial_s|\widetilde{\mathbf{F}}|^2$ is strictly negative whenever $|\widetilde{\mathbf{F}}|>C_0$. Let

$$\Lambda := \max\left\{C_0, \max_{\widetilde{M}(s_0)}|\widetilde{\mathbf{F}}|\right\}.$$

Thus, going back in time, $\widetilde{M}(s)$ cannot escape the ball $B_{2\Lambda}(0)$. □

Corollary 3.6. *Let M be a type-I mean curvature flow and $p \in \mathcal{M}$. Then $\widetilde{M}^{p^*}(s) \cap B_{2C_0}(0)$ is nonempty, for every $s \in \mathbb{R}$.*

In particular, due to Lemma 3.4, Corollary 3.6 applies to rescaled flows centered at singular points.

Proof. By Lemma 3.2,

$$|\widetilde{\mathbf{F}}^{p^*}(p,s)| = |\lambda(t)\left(\mathbf{F}(p,t) - p^*\right)| \leq \lambda(t) C_0 \lambda^{-1}(t) = C_0$$

for all $s \in \mathbb{R}$. Then $\widetilde{\mathbf{F}}^{p^*}(p,s) \in B_{2C_0}(0)$ for all $s \in \mathbb{R}$. Therefore $\mathcal{M}(s)$ intersects $B_{2C_0}(0)$ for every $s \in \mathbb{R}$. $\square$

Note the proof is with regards to the p corresponding to p^*, not $|\widetilde{\mathbf{F}}|$ globally.

Chapter 4

Type-I Singularities

4.1 Some Background on Blow-ups

In §2.2 we introduced two common blow-up techniques, the rescaled flow and tangent flows, and we use both. For our purposes, there are stronger results for tangent flows, such as uniqueness against subsequence. On the other hand, the rescaled flow is convenient since it only deals with one flow, and certain objects remain stationary. Luckily, the calculation in (2.4) shows that any sequence of times $s_i \nearrow \infty$ in the rescaled flow corresponds to a particular tangent flow.

White showed in [43] that, in the mean-convex case, all tangent flows are either planes, (generalized) cylinders, or spheres. Furthermore, we have from [40] that planes are ruled out for times approaching the first singular time from below. Our main result is for $N = 2$, so we deal mostly with cylinders as blow-up limits. These limits, however, are those of subsequences. The question arises whether the limit depends on the subsequence. That is, the cylinder shape and radius are fixed, but can the orientation change per subsequence? Colding and Minicozzi find in [12] that it cannot: If one tangent flow is a cylinder, then they all are. In fact, they are all the same cylinder.

What about the rescaled flow? Huisken showed in [28] that the rescaled flow, when centered around a singular point, converges smoothly on compact subsets to a stationary limit. This corresponds to a self-similar flow in the nonrescaled setting, which we know from [28] and [29] indicates a cylinder or sphere. So if we can connect this notion of rescaling to tangent flows, we can control limits of $\widetilde{M}(s)$, because of the uniqueness of cylindrical tangent flows.

4.2 A Compactness Theorem for Hypersurfaces

4.2.1 Background

There are already a number of types of convergence and compactness results for hypersurfaces, from Cheeger-Gromov compactness for metrics to C^1 compactness for parameterizations. Endowing hypersurfaces with a type of non-collapsing behavior, our compactness here is in terms of local graph-like convergence on compact sets. This is slightly stronger in the local sense that in small neighborhoods, only a single component of each hypersurface is present. This allows for much better descriptions of convergence in neighborhoods.

The original motivation for this specific type of convergence lies in the study of blow-ups for geometric flows. Specifically, Huisken finds in [28] that for compact mean curvature flows, with type-I curvature control, blow-ups at singularities converge smoothly to limit hypersurfaces. Furthermore, for mean-convex hypersurfaces, he (and White in [43]) find that blow-ups are generalized cylinders, spheres, or hyperplanes.

However, his method relies closely on one of Langer from [35] showing C^1 convergence of immersions. This means that Huisken's limit is merely an immersed submanifold, and the proof is per component of the immersions. This allows for a number of issues such as self-intersection, or from the measure-theoretic viewpoint, multiplicity. In the author's case, there was the need to express hypersurfaces in a converging sequence as graphs of functions over the limit hypersurface (a known generalized cylinder) for precise analysis, which is not possible unless intrinsically far regions of the hypersurface are kept apart extrinsically.

Enter Andrews' non-collapsing condition [2]. Given a bound on curvature, the condition prevents intrinsically far points from becoming extrinsically close. Andrews shows that the condition is uniformly preserved by mean curvature flow, so it is a natural candidate for the extra needed hypothesis. Unfortunately, the condition is only defined for strictly mean-convex hypersurfaces. So instead we use the condition of uniformly positive reach (introduced by Federer in [17] and described in the survey [42]). For mean-convex hypersurfaces , the reach condition is equivalent to Andrews non-collapsing condition with bounded curvature. However, reach is defined for any closed subset of a metric space.

Our main theorem follows more or less directly from a theorem of Breuning in [8], which follows closely the result of Langer in [35]. The main caveat is that it is not obvious that our hypotheses meet the local area bound required for Breuning's theorem. We show that our upper bound on curvature and lower bound on reach imply the necessary area bound. This is explained in a little more detail after we state Breuning's theorem (here Theorem 4.10).

Graph convergence has the added bonus that the sequence hypersurfaces can be expressed as graphs of single functions over the entire limit hypersurface (in arbitrarily large compact sets). In the case of mean curvature flow, Colding and Minicozzi show in [12] that the cylindrical limits are unique, independent of sequence of times shown. Together with the main result herein, that means that the blown-up flow can be expressed as the graph of a single evolving function over truncated subsets of the cylinder.

4.2.2 Convergence Result

Before stating our main result, we must precisely define the type of convergence we mean. This is a noncompact version of our previously defined graph convergence.

Definition 4.1 (Graph Convergence)**.** *Assume $k \geq 1$. Let Σ and Σ_n be C^k-smooth, properly embedded hypersurfaces in $\mathbb{R}^{N+1}$. Assume Σ is oriented by a smooth normal vector field ν. We say Σ_n converges to Σ locally in the graph sense to order k if the following holds:*

For every open ball $B \subset \mathbb{R}^{N+1}$, there is $n_0 > 0$ so that whenever $n \geq n_0$

i) The limit set $\Sigma \cap B$ is the set of all accumulation points of Σ_n in B. That is, $\Sigma \cap B$ is the set of all $x \in B$ such that there is a sequence of points $x_n \in M_n$ with $x_n \to x$.

ii) If $\Sigma \cap B$ is nonempty, the nearest point map

$$\pi_n^B : \Sigma_n \cap B \to \Sigma$$

is a well-defined diffeomorphism onto its image $V_n^B \subset \Sigma$.

iii) For $y \in M_n \cap B$, write $x = \pi_n^B(y)$. Then define $g_n^B : V_n^B \to \mathbb{R}$ to be the height function

$$g_n^B(x) = (y - x) \cdot \nu(x),\ \ y = (\pi_n^B)^{-1}(x)$$

over $V_n^B \subset \Sigma$, so that for all $x \in V_n^B$

$$(\pi_n^B)^{-1}(x) = x + g_n^B(x)\nu(x).$$

Thus g_n^B is the signed height of $\Sigma_n \cap B$ over $V_n^B \cap \Sigma$. Then

$$\|g_n^B\|_{C^k(V_n^B)} \xrightarrow[n\to\infty]{} 0.$$

Remark 4.2. *Thus for n sufficiently large, $\Sigma_n \cap B$ is a normal graph over $V_n^B \subset \Sigma$, and we call this* local graph convergence*. For this we may write $\Sigma_n \xrightarrow{C^k_{loc}} \Sigma$ or just $\Sigma_n \to \Sigma$.*

Notice that the domain of g_n^B may not coincide with $\Sigma \cap B$. But we can just choose a larger n and larger ball B, so convergence will behave as one would expect on any fixed compact set contained in the open ball B.

Remark 4.3. *No topological information about Σ_n, other than dimension, is preserved in the limit. For example, the limit Σ could be noncompact, while the Σ_n are compact.*

There is one more condition that needs defining before the main statement.

Definition 4.4 (Reach)**.** *Let $X \subset \mathbb{R}^{N+1}$ be closed and let X_r be the r-neighborhood of X. Define the* reach *of a set $X \subset \mathbb{R}^{N+1}$ to be*

$$\operatorname{reach} X := \sup\{r \geq 0 : \textit{ for all } y \in X_r, \exists\, !\, x \in X \textit{ closest to } y\}.$$

For a smooth hypersurface M, the reach $\mathcal{R}$ can be thought of as preventing M from intersecting any of its own normals up to distance $2\mathcal{R}$ away. For smooth hypersurfaces, *reach* is also known as the normal injectivity radius.

Theorem 4.5 (Main Theorem)**.** *Let $M_n \subset \mathbb{R}^{N+1}$ be a sequence of $(k+2)$-smooth, properly embedded, complete hypersurfaces with* $\operatorname{reach} M_n \geq \mathcal{R} > 0$ *and $|\nabla^m A_n| \leq C_k$ for $0 \leq m \leq k$. Assume, for some closed ball B centered at 0, that $M_n \cap B$ is nonempty for all n.*

Then there is a $(k+1)$-smooth, properly embedded, complete hypersurface M such that a subsequence of M_n converges to M locally in the graph sense to order $(k+1)$ on compact sets containing B. Moreover, $\operatorname{reach} M \geq \mathcal{R}$, *$|\nabla^m A_M| \leq C_k$ for $0 \leq m \leq k$, and M intersects B.*

Remark 4.6. *We will find in Proposition 4.14 that the lower bound on reach implies M is properly embedded anyway. Of course proper embedding implies completeness for manifolds without boundary. However, the theorem is more digestible in this form.*

In fact, for a hypersurface with positive mean curvature, having positive reach is equivalent to the Andrews non-collapsing condition. We show in Lemmas 4.8 and 4.9 that the non-collapsing condition with a fixed α is equivalent to a lower bound on reach (in the context of Theorem 4.5). Thus we have the following corollary.

Corollary 4.7. *The main theorem holds with "*$\operatorname{reach} M_n \geq \mathcal{R} > 0$*" replaced by "each M_n is α-non-collapsed" for some $\alpha > 0$.*

4.3 Preliminaries

4.3.1 Notation

- $\subset$: always means strict inclusion.
- ι: identity map
- $B_r = B_r^{N+1}(0)$
- $\widehat{R} = \{x_{N+1} = 0\}$: hyperplane
- $\widehat{B}_r = B_r \cap \widehat{R}$
- e_i: unit vector in the x_i direction
- $\nu(x)$: normal to a hypersurface with chosen orientation

4.3.2 Reach and Non-collapsing

Lemma 4.8. *Let $\Sigma \subset \mathbb{R}^{N+1}$ be a smooth, mean-convex hypersurface. Assume $|A| \leq C_0$ and Σ is α-non-collapsed. Then* $\operatorname{reach} \Sigma \geq \mathcal{R} > 0$, *where $\mathcal{R}$ depends only on C_0 and α.*

Proof. Choose $\mathcal{R} := \dfrac{\alpha}{\sqrt{N} C_0}$. Observe that at every point of Σ,

$$\mathcal{R} \leq \frac{\alpha}{\sqrt{N}|A|} \leq \frac{\alpha}{H}.$$

Then, since Σ is α-non-collapsed, at each point $p \in \Sigma$, there is an open ball of radius $\mathcal{R}$, tangent to Σ at p, on either side of Σ, such that Σ does not intersect either ball.

Suppose $\operatorname{reach} \Sigma < \mathcal{R}$. Then there are $q \in \mathbb{R}^{N+1}$ and distinct points $p_1, p_2 \in \Sigma$, which minimize $\operatorname{dist}(q, \Sigma)$, such that

$$|q - p_1| = |q - p_2| < \mathcal{R}.$$

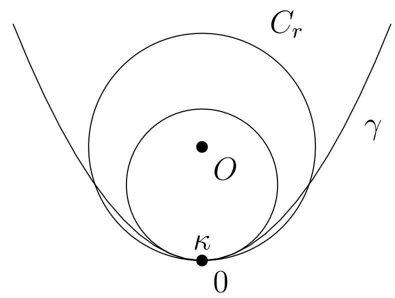

Figure 4.1: C_r hangs slightly lower than γ near 0.

Therefore there is a sphere S, centered at q and tangent at p_1 and p_2. However, since the radius of S is less than $\mathcal{R}$, we must have that p_2 is contained in one of the two $\mathcal{R}$-balls tangent to Σ at p_1. But we already found Σ cannot intersect that ball, so we have a contradiction.

$\square$

Lemma 4.9. *Let $\Sigma \subset \mathbb{R}^{N+1}$ be a smooth, mean-convex hypersurface. Assume* $\operatorname{reach}\Sigma \geq \mathcal{R} > 0$. *Then $|A| \leq C_0$ and Σ is α-non-collapsed, where C_0 and α depend only on $\mathcal{R}$.*

Proof. **Step One:** Curvature is bounded.

Take $p \in \Sigma$. Assume, without loss of generality, that $p = 0$ and the normal $\nu(p) = \nu(0) = e_{N+1}$. Let P be any 2-plane containing $p = 0$ and $\nu(p) = e_{N+1}$ and consider $\gamma = \Sigma \cap P$. Let κ be the curvature of γ at 0 (see Figure 4.1).

Take $r = \frac{\mathcal{R}+\frac{1}{\kappa}}{2}$. Let C_r be the circle in P, with radius r and tangent to γ at 0. Let q be its center. Near 0, γ and C_r are graphs over some interval containing 0 of functions, say, f and g, respectively. Note

$$\kappa = \frac{f''(0)}{(1+|f'(0)|^2)^{\frac{3}{2}}} = f''(0).$$

Suppose $\kappa > \frac{1}{\mathcal{R}}$. Then $g''(0) < f''(0)$, where we assumed

$$f(0) = f'(0) = g(0) = g'(0) = 0,$$

so that $g \leq f$ on some interval around 0.

If C_r intersects γ at more than one point, then q is equidistant to more than one point in γ, which contradicts the reach hypothesis, since $r < \mathcal{R}$. Thus $\kappa \leq \frac{1}{\mathcal{R}}$. However, if not, consider if C_r were to shrink, but remain tangent at 0. Once the radius is less than $\frac{1}{\kappa}$, the new circle is above γ over a deleted neighborhood of 0. Therefore there was some radius were the circle crossed over γ. We can apply the same argument to that circle, since its radius is between $\frac{1}{\kappa}$ and $\mathcal{R}$. Thus again $\kappa \leq \frac{1}{\mathcal{R}}$.

We are left to conclude $|A| \leq C_0 := \frac{\sqrt{N}}{\mathcal{R}}$, since

$$|A|^2 = \sum_i \kappa_i^2 \leq N \frac{1}{\mathcal{R}^2},$$

where the κ_i are the principle curvatures of Σ at $p = 0$.

Step Two: Σ is α-non-collapsed.

Suppose for some $p_1 \in \Sigma$, either open ball of radius $r < \mathcal{R}$, tangent to Σ at p_1 contains a point $p_2 \in \Sigma$ distinct from p_1. But then the sphere that is the ball's boundary intersects Σ at more than one point, so its center is equidistant to more than one point of Σ, meaning reach $\Sigma \leq r < \mathcal{R}$, which is a contradiction. Thus if $\alpha := \sqrt{N} C_0 \mathcal{R}$, we have

$$\mathcal{R} = \frac{\alpha}{\sqrt{N} C_0} \leq \frac{\alpha}{H}.$$

Therefore Σ is α-non-collapsed.

□

4.3.3 Reduction to Breuning's Theorem

Our proof uses Breuning's Theorem 1.3 of [8], so we state that here. However, since we are dealing with hypersurfaces, we save some notation in assuming his parameterizations are inclusion maps. That is, where Breuning assumes a sequence of proper immersions f^i, we assume proper embeddings f_n. Denote the N-dimensional Hausdorff measure by $\mathcal{H}^N$, and the identity map by ι. Note that in the following, although n indexes the sequence of hypersurfaces, B_i refers to the ball centered at the origin with radius n.

Theorem 4.10 (Breuning)**.** *Let $f^i : M^i \to \mathbb{R}^{N+1}$ be a sequence of proper embeddings, where M^i is an N-manifold without boundary, and $f^i(M^i) \cap B \neq \varnothing$ for some ball B centered at 0. Assume*

$$\begin{aligned}\mathcal{H}^N(f^i(M^i) \cap B_R) &\leq C(R) \textit{ for all } R > 0, \textit{ and} \\ \|\nabla^k A^i\|_{L^\infty(B_R)} &\leq C_k(R) \textit{ for all } R > 0 \textit{ and } k \in \mathbb{N}.\end{aligned}$$

Then there exists a proper immersion $f : M \to \mathbb{R}^{N+1}$, where M is again an N-manifold without boundary, such that after passing to a subsequence, there are diffeomorphisms

$$\phi^i : U^i \to (f^i)^{-1}(B_i) \subset M^i,$$

where $U^i \subset M$ are open sets with $U^i \subset\subset U_{n+1}$ and $M = \bigcup_{n=1}^\infty U^i$, such that $\|f^i \circ \phi^i - f\|_{C^0(U^i)} \to 0$, and moreover $f^i \circ \phi^i \to f$ locally smoothly on M.

Moreover, M also satisfies.

$$\begin{aligned}\mathcal{H}^N(f(M) \cap B_R) &\leq C(R) \textit{ for all } R > 0, \textit{ and} \\ \|\nabla^k A\|_{L^\infty(B_R)} &\leq C_k(R) \textit{ for all } R > 0, \textit{ and } k \in \mathbb{N}.\end{aligned}$$

The area bound is the only one of Breuning's hypotheses that is not obviously satisfied by the hypersurfaces in our theorem. So our primary task is to show that our hypotheses do in fact meet that criterion.

Recall that we denote $\widehat{R} = \{x_{N+1} = 0\}$ so that we can write $B_r = B_r^{N+1}(0)$, and $\widehat{B}_r = B_r \cap \widehat{R}$. The main hypotheses in our theorem are the curvature and reach bounds. The plan is that if a hypersurface (after translation and rotation) is locally a graph over $\widehat{B}_r$

for some small r, then we can use the reach bound to maintain a minimum vertical distance between the graph and other parts of the hypersurface, which could otherwise prevent the graph's function from being well-defined. We can use the curvature bound to maintain a minimum radius for $\widehat{B}_r$ on which we have a uniform bound on the derivative of the graph, and therefore of its area. (This is the reverse meaning of B and $\widehat{B}$ in [8].)

Combining the lower bound on the radius of $\widehat{B}_r$, the well-defined function for the graph, and the curvature (and therefore gradient) bound, we can obtain a minimum neighborhood over which the graph has uniformly bounded area. That is, the bound depends only on the curvature bound, the reach, and the radius of $\widehat{B}_r$. Furthermore, for any given compact set, the minimum bound on the neighborhood size means there is a maximum number of such neighborhoods to cover the compact set. Thus, we can find an area bound for $M_n \cap B_R$, dependent only on R.

We prove a simplified version of Theorem 2.6 of [8] (similar to Theorem 2.4 of [35]), showing the desired derivative bound.

Lemma 4.11. *Let Σ be a C^2-embedded, complete hypersurface with $|A| \leq C_0$. Further assume $0 \in \Sigma$ and $\nu(0) = e_{N+1}$ and define*

$$\mathcal{C}_r := \widehat{B}_r \times [-r, r] \subset \mathbb{R}^{N+1}.$$

Then there is $r_1 = r_1(C_0) > 0$ such that the 0-component of $\Sigma \cap \mathcal{C}_{r_1}$ is the graph of a function u over $\widehat{B}_r$ with

$$\|Du\|_{C^0(\widehat{B}_r)} \leq 2C_0 r$$

for $r \leq r_1 = \frac{1}{2C_0}$.

Proof. Since Σ is embedded, we know it is locally a graph over is own tangent plane. Therefore, there is some $\rho > 0$ such that the 0-component of $\Sigma \cap (\widehat{B}_\rho \times \mathbb{R})$ is the graph of some function u over $\widehat{B}_\rho$ with $\|Du\|_{C^0} \leq \frac{1}{2}$.

For any $r > 0$ such that $\|Du\|_{C^0(\widehat{B}_r)} < \infty$, the final calculation in Lemma 2.2 of [8] yields

$$|\partial_{ij}u| \leq (1 + \|Du\|_{C^0(\widehat{B}_r)})^{\frac{3}{2}}|A|.$$

Since $|A| \leq C_0$, that means

$$\|D^2u\|_{C^0(\widehat{B}_r)} \leq NC_0 \left(1 + \|Du\|_{C^0(\widehat{B}_r)}\right)^{\frac{3}{2}}.$$

By the Mean Value Inequality, we have for $r \leq \rho$

$$\|Du\|_{C^0(\widehat{B}_r)} \leq r\|D^2u\|_{C^0(\widehat{B}_r)}.$$

Putting together the previous two inequalities, we have

$$\|Du\|_{C^0(\widehat{B}_r)} \leq NC_0 r \left(1 + \|Du\|^2_{C^0(\widehat{B}_r)}\right)^{\frac{3}{2}}. \tag{4.1}$$

Of course, this suggests we can extend the domain for u while $|Du|$ is small. Because Σ is C^2 and complete, for any r such that $\|Du\|_{C^0(\widehat{B}_r)} < \infty$, u can be extended to $Cl(\widehat{B}_r)$. Similarly, whenever $\|Du\|_{C^0(Cl(\widehat{B}_r))} < \infty$, u can be extended to $\widehat{B}_{r+\varepsilon}$, for some $\varepsilon > 0$. Therefore we may extend the domain of u until $\|Du\|_{C^0(\widehat{B}_r)} = 1$.

For simpler notation and calculation, let us write $a_r := \|Du\|_{C^0(\widehat{B}_r)}$, and $\rho_0 = \max\{r : a_r \leq 1\}$ (if this does not exist, then we are already done with the proof). Then on $\widehat{B}_{\rho_0}$, we have (by Jensen's Inequality)

$$a_r \leq C_0 r(1 + a_r^2)^{\frac{3}{2}} \leq \frac{C_0}{\sqrt{2}} r(1 + a_r^3) \leq 2C_0 r.$$

Intuitively, this means if r is very small (and at least as small as ρ_0), we should be able to grow it to a fixed minimum size where a_r is small enough that the above estimates still hold. In that light, we claim that $\rho_0 \geq \frac{1}{2C_0}$. Suppose instead that $\rho_0 < \frac{1}{2C_0}$. Then

$$1 = a_{\rho_0} \leq 2C_0\rho_0 < 1,$$

which is clearly a contradiction. Thus, whenever $r \leq \frac{1}{2C_0}$, $\|Du\|_{C^0(\widehat{B}_r)} \leq 1$.

Finally, let $r_1 = \frac{1}{2C_0} \leq \rho_0$ so that $\|Du\|_{C^0(\widehat{B}_r)} < 1$ for any $r < \rho_0$. We have again by the Fundamental Theorem of Calculus,

$$\|u\|_{C^0(\widehat{B}_r)} \leq r\|Du\|_{C^0(\widehat{B}_r)} \leq 2C_0 r^2 < \frac{1}{2C_0} = r_1.$$

So there are no worries about whether the 0-component of $\Sigma \cap \mathcal{C}$ is a graph over $\widehat{B}_{r_1}$, since truncating the cylinder on top and bottom will not truncate the graph.

□

Lemma 4.12. *Let Σ be as in Lemma 4.11 with the added assumption that* $\operatorname{reach} \Sigma \geq \mathcal{R} > 0$. *Let $r_2 = \min\{r_1, \mathcal{R}\}$.*

Then all of $\Sigma \cap \mathcal{C}_{r_2}$ is the graph of a function u over $\widehat{B}(r_2)$.

That is, not just the 0-component. Note by Lemma 4.9, it is possible to remove the curvature bound from the hypotheses and just take $r_2 = \frac{\mathcal{R}}{\sqrt{2}} \leq r_1$.

Proof. Suppose Σ_1 and Σ_2 are two distinct components of $\Sigma \cap B_{2\mathcal{R}}$. Without loss of generality, say Σ_1 is the component containing 0. Since $B_{2\mathcal{R}}$ is convex, there is a point $q \in B_{2\mathcal{R}}$ that is equidistant to Σ_1 and Σ_2. Since $0 \in \Sigma_1$ and $\Sigma_2 \cap B_{2\mathcal{R}} \neq \varnothing$, $\operatorname{dist}(\Sigma_1, \Sigma_2) < 2\mathcal{R}$. Then $\operatorname{dist}(q, \Sigma_1) < \mathcal{R}$. But that contradicts the assumption that $\operatorname{reach} \Sigma \geq \mathcal{R}$!

Now apply Lemma 4.11, and we are done. □

One can see intuitively why this should be true by drawing two U-shaped curves, one very shallow and one very sharp. Next draw the r-neighborhood for each, where r is small compared to the shallow turn, but approximately the radius of the sharp turn. The former produces a long tube, while the latter produces a semi-infinite cigar shape. In both cases, an $\frac{r}{2}$-ball centered at the tip of the turn fits well within the r-neighborhood.

Corollary 4.13. *Let Σ and r_2 be as in Lemma 4.12. For each $p \in \Sigma$, we have*

$$\mathcal{H}^N\left(B_{r_2}(p) \cap \Sigma\right) \leq 2\omega_N r_2^N,$$

where ω_N is the volume of the unit N-ball.

Proof. Assume, without loss of generality, that $p = 0$, and the normal $\nu(p) = \nu(0) = e_{N+1}$. Of course Lemma 4.11 and Lemma 4.12 apply, so $\Sigma \cap \mathcal{C}_{r_2}$ is the graph of some function u over $\widehat{B}_{r_2}$, with $u(0) = 0, Du(0) = 0$, and $\|Du\|_{C^0(\widehat{B}_{r_2})} \leq 1$.

Then

$$\sqrt{1 + |Du|^2} \leq 1 + |Du| \leq 2,$$

on $\widehat{B}_{r_2}$, so we calculate

$$\mathcal{H}^N(\mathcal{C}_{r_2} \cap \Sigma) = \int_{\widehat{B}_{r_2}} \sqrt{1 + |Du|^2}\, dx \leq 2\omega_n r_2^N.$$

Of course, the same estimate holds for B_{r_2}, since it is contained in $\mathcal{C}_{r_2}$. Since p was arbitrary, this bound holds throughout Σ. □

Proposition 4.14. *Let Σ be as in Lemma 4.12. For every $R > 0$, there is $C(R) = C_{C_0,\mathcal{R}}(R)$ so that $\mathcal{H}^N(B_R \cap \Sigma) \leq C(R)$.*

Proof. Let $r_0 = \frac{r_2}{N} < \frac{r_2}{\sqrt{N}}$ so that it is clear that $K_0 := [r_0, r_0]^{N+1} \subset B_{r_2}$. Then write $K_0 + p$ for K_0 shifted to be centered at p. That way for $p \in \Sigma$, $\Sigma \cap (K_0 + p)$ satisfies the same area bound as $\Sigma \cap B_{r_2}(p)$.

Since the side length of K_0 is $2r_0$, we know that the box $K(R) = [-R, R]^{N+1}$ can be covered by

$$\mathcal{N} := \left\lceil 2\frac{R}{2r_0} \right\rceil^{N+1} = \left\lceil \frac{R}{r_0} \right\rceil^{N+1}$$

boxes congruent to K_0. Therefore we have by Corollary 4.13 that

$$\mathcal{H}^N(\Sigma \cap K(R)) \leq 2\omega_N r_2^N \mathcal{N}$$

Certainly $B_R \subset K(R)$, so we are done. □

We therefore can apply Breuning's theorem! Breuning's theorem does require that the f_n are C^∞, but the method in Step 6 of his main proof obtaining higher order convergence, only uses the k-th order bound on A for the $(k+2)$-th order bound on local graph derivatives, so we really only need a finite-order bound to get finite-order convergence.

4.3.4 Proof of the Main Theorem

Assume all the hypotheses of the main theorem. We now know the hypotheses of Breuning's theorem are met. Furthermore, since we assume the M_n are embedded hypersurfaces, we can take the f_n in Breuning's theorem to be inclusion maps. That is, we have that $\phi_n \xrightarrow{C^k_{loc}(M)} f$ for any $k \in \mathbb{N}$.

From here we need to:

- show M and f have the desired properties ,
- establish the ϕ_n are indeed the maps we want ,
- show our definition of convergence is satisfied .

Properties of M

We first want to show that, under our hypotheses, the limit hypersurface M obtained by Breuning's theorem is smoothly embedded. This is part of our theorem of course, but it also allows us to apply previous results and generally makes discussion of the map f easier.

Lemma 4.15. *The f given by Breuning's theorem is a proper smooth embedding.*

Proof. We already have from Breuning that f is a proper immersion. So we need only show injectivity. Suppose $f(p) = f(q)$ for distinct $p, q \in M$. Assume without loss of generality that $f(p) = f(q) = 0$ and that $df(p)(T_p) = \widehat{R}$. That is, the plane tangent to $f(M)$ at $f(p) = 0$ is horizontal.

We know that $\phi_n(p), \phi_n(q) \to 0$ and $d\phi_n(p), d\phi_n(q) \to df(p)$. Thus Lemma 4.12 tells us for large enough n, there is a radius r such that the 0-component of $M_n \cap (\widehat{B}_r(0) \times \mathbb{R})$, call it Σ_n, is a graph of some function u_n over $\widehat{B}_r(0) \subset \widehat{R} = df(p)(T_pM)$, with $\|Du\|_{C^0(\widehat{B}_r(0)} \leq 1$. Then Σ_n is indeed a hypersurface. From here the basic idea is to use that Σ_n is embedded to know that as $|\phi_n(q) - \phi_n(p)| \to 0$, we must have that $\mathrm{dist}_{\Sigma_n}(\phi_n(q), \phi_n(p)) \to 0$. This will imply that $\det(d\phi_n)$ is collapsing somewhere, so $\det(df = 0)$ somewhere, which contradicts that f is an immersion.

As stated, the first step is to verify extrinsic closeness in Σ_n implies intrinsic closeness. Write $x_n = \phi_n(p), y_n = \phi_n(q)$ and let $\widehat{x}_n = \pi(x_n), \widehat{y}_n = \pi(y_n)$ be their projections onto $\widehat{R}$. We have the following three facts: $|\widehat{y}_n - \widehat{x}_n| \leq |y_n - x_n|$,

$$\sqrt{1 + |Du|^2} \leq 1 + |Du| \leq 2,$$

and any curve in Σ_n from x_n to y_n will have length at least $\mathrm{dist}_{\Sigma_n}(x_n, y_n)$. Together, they give us

$$\mathrm{dist}_{\Sigma_n}(x_n, y_n) \leq \int_0^1 \sqrt{1 + |Du((1-t)\widehat{x}_n + t\widehat{y}_n)|^2} \, (|\widehat{y}_n - \widehat{x}_n| \, dt) \leq 2|x_n - y_n|.$$

Finally, we know f is a smooth diffeomorphism because of the smooth convergence of ϕ_n to f. □

Now since we know f is a smooth embedding, we might as well assume it is the inclusion map and that M is a submanifold of $\mathbb{R}^{N+1}$.

Proposition 4.16. *The M obtained by Theorem 4.10 satisfies the geometric properties of the main theorem.*

Proof. Since f and f_n are the inclusion maps, the theorem gives us an M that is $(k+1)$-smooth, properly embedded, complete hypersurface with $|\nabla^m A| \leq C_k$. Of course $M \cap Cl(B)$ by continuity.

What's left to show of the properties of M is reach $M \geq \mathcal{R}$. Suppose it is not. Then there are $q \in \mathbb{R}^{N+1}$ and $p_1, p_2 \in M$ such that

$$|q - p_1| = |q - p_2| < \mathcal{R}.$$

But then for some n large enough, there are $p_3, p_4 \in M_n$ such that

$$|q - p_3| < \frac{\mathcal{R} + |q - p_1|}{2} < \mathcal{R},$$

and similarly $|p_4 - q| < \mathcal{R}$. Recalling $|q - p_1| = |q - p_2|$, set $\rho := \frac{\mathcal{R}+|q-p_1|}{2} < \mathcal{R}$.

We know $p_3, p_4 \in B_\rho(q)$. Either $\partial B_\rho(q)$ intersects M_n at more than one point, or M_n is compact and $M_n \subset B_\rho(q)$, in which case we can choose $r \leq \rho$ so that $\partial B_r(q)$ intersects M_n at more than one point. Both cases contradict the assumption that reach $M_n \geq \mathcal{R}$. $\square$

Normal Projection

Our desired type of convergence now follows relatively naturally from the statement of Breuning's theorem with the exception that the reparameterizations ϕ^i are not explicitly given. It turns out in the proof of Theorem 4.10 the ϕ^i are indeed defined to be the inverse normal projections we need! However, since Breuning deals with immersions, the situation in [8] is more complicated. In order to understand the definition there, we must walk through a few constructions. We deal with the compact case first.

Remember in the context of Breuning's theorem, M and the M^i are abstract manifolds, and f and the f^i merely immersions into $\mathbb{R}^{N+1}$. Recall we use the notation $B_r \subset \mathbb{R}^{N+1}$, $\widehat{B}_r \subset \mathbb{R}^N$. (Breuning uses the reverse notation in the noncompact case.)

Following §2 of [8], begin with covers of the M^i in order to describe a system of graphs on them. For any q in an abstract manifold M and an immersion $f : M \to \mathbb{R}^{N+1}$, let ρA_q be an isometry on $\mathbb{R}^{N+1}$ that maps 0 to $f(q)$ and takes $\widehat{R}$ to the tangent space $df(T_qM)$ (so that ρA_q

takes e_{N+1} to a normal vector to $f(M)$ at $f(q)$). Then, recalling r_2 from our Lemma 4.12, for $r \in (0, r_2)$ take $U_{r,q} \subset M$ to be the q-component of $(\pi \circ (\rho A_q)^{-1} \circ f)^{-1}(\widehat{B}_r)$. We also know by Lemma 4.12 that $f(U_{r,q})$ can be represented as the graph of a function u_q over $\widehat{B}_r$. By Lemma 4.11, for each $\alpha \in (0,1)$, there is an $r \in (0, r_2)$ such that $\|Du_q\|_{C^0(\widehat{B}_r)} \leq \alpha$, with r dependent only on α and C_0.

In the compact case, choose finite sets $Q^i = \{q^i_j\}_{j=1}^s \subset M^i$ such that, for some $\delta > 0$, the collection of U^i_{δ, q^i_j} covers M^i (the volume bound allows for a uniform bound in s). Thus the $u^i_{q_j}$ create a system of graphs representing $f^i(M^i)$. Convergence of immersions equipped with graph systems (Definition 3.1 of [8]) corresponds to C^k convergence of the functions $u^i_{q_j}$ over $\widehat{B}_\delta$, uniform in j. To each q^i_j is associated $\widehat{B}^j_\delta = \widehat{B}_\delta \times \{j\}$, over which $u^i_{q_j}$ is a graph (this is independent of i since the graph systems converge in i). The limit manifold M is constructed from the disjoint union of the $\widehat{B}^j_\delta$, so $f(M)$ is represented by the system of graphs of functions u_j. Then each $\left((\rho A^i_{q_j})^{-1} \circ f^i\right)\left(U^i_{\delta, q^i_j}\right)$ is close to $\widehat{B}_\delta$, so the inverse normal projection definition defined below makes sense. So for points $(x, j) \in B^j_\delta \times \{j\}$, it is natural to compare the maps $f(x)$ and $\phi^i_j \circ f^i$, where ϕ_{j^i} are diffeomorphisms from B^j_δ to U^i_{δ, q_j} (radii have to be adjusted slightly). The ϕ^i_j can be patched together for the following definition.

The definition of the ϕ^i can be found in the discussion after Lemma 5.2 in [8]. Let $x \in M$ (Breuning sometimes conflates $\widehat{B}^j_\delta$ and $\widehat{B}_\delta$). Then $x \in P(\widehat{B}^j_\delta)$ for some j. Let $h(x)$ be the affine subspace $f(x) + \mathrm{span}\{\nu(f(x))\}$ normal to $f(M)$ at $f(x)$. Assume $\alpha^2 \leq \frac{1}{10}$, choose r accordingly and let $\delta = \frac{r}{16}$, and assume n is large. Then Lemma 5.1 of [8] says that $h(x)$ intersects $f^i(U^i_{r,j}) \subset f^i(M^i)$ at exactly one point, call it S_x. There is of course exactly one $\sigma_x \in U^i_{r,j} \subset M^i$ with $f^i(\sigma_x) = S_x$. Finally, Lemma 5.2 of [8] gives that $\phi^i(x) = \sigma_x$ is a well-defined map, despite the different neighborhoods $U^i_{r,j}$ in the cover. Then $(\phi^i)^{-1}$ is the inverse normal projection!

In the noncompact case, the diffeomorphisms are constructed locally in a similar way, but a sort of diagonal argument is used to achieve global convergence on compact sets.

Showing Convergence

i) Let $x \in M$. Then $x_n := \phi_n(x) \to x$.

Now assume x is an accumulation point of M_n. So there are $x_n \in M_n$ with $x_n \to x$. Since $\{x_n\}$ is bounded, $x_n \in B_n$ for large enough n. So since f_n is the inclusion map,

$$x_n \in M_n \cap B_n = \phi_n(U_n).$$

Of course, $\phi_n \rightrightarrows \iota$, so x_n must converge to a point of M.

ii) For any ball $B \subset \mathbb{R}^{N+1}$, there is large enough n such that

$$V_n^B \subset V_n^{B_n} := \pi_n^{B_n}(B_n) = \phi_n^{-1}(B_n) = U_n,$$

so $\pi_n^B = \phi_n^{-1}$ is a diffeomorphism on $M_n \cap B$.

iii) We can fix some ball B to suppress it in the notation in the following calculations.

$$\begin{aligned} \phi_n &= \iota + g_n \nu \\ D\phi_n &= I + Dg_n \otimes \nu + g_n D\nu \\ D^k \phi_n &= D^k g_n \otimes \nu + \textstyle\sum_{l=1}^k \binom{k}{l} D^{k-l} g_n \otimes D^l \nu, \text{ for } k \geq 2 \ . \end{aligned}$$

Then since $\phi_n \rightrightarrows \iota$, $g_n \rightrightarrows 0$. Similarly, since $D\phi_n \rightrightarrows I$ and ν does not depend on n, we have $Dg_n \rightrightarrows 0$. Then we can continue by induction to show that any derivative of g_n, past the first, converges uniformly to 0 on its domain.

Now we have established that $M_n \to M$ in the local graph sense.

4.4 Classifying Singularities

Lemma 4.17. *Let M be a smoothly embedded, closed, mean-convex, type-I mean curvature flow with a singular point 0 at time T. Then for any sequence of rescaled times $s_i \nearrow \infty$, there is a subsequence $\sigma = \{s_{i_j}\}$ for which $\widetilde{M}(s_{i_j})$ converge to some $\widetilde{M}_\infty^\sigma$ in the graph sense, where $\widetilde{M}_\infty^\sigma$ meets all the criteria of M in the conclusion of Definition 4.1.*

Proof. We simply need to show that the $\widetilde{M}(s_i)$ satisfy the hypotheses of Definition 4.1. We already have that they are smooth and embedded. Since $M(t)$, and therefore $\widetilde{M}(s)$, is closed, we know the $\widetilde{M}(s_i)$ are complete and are properly embedded. Since M is compact and type-I, Proposition 2.3 of [28] directly gives us the curvature bounds. Our Lemma 3.2 tells us that $\widetilde{M}(s_i) \cap B_{2C_0}(0)$ is nonempty.

Finally, we must show the $\widetilde{M}(s_i)$ are α-non-collapsed (that is, α is independent of n). Since $M(t)$ is mean-convex, by the strong minimum principle, it is strictly non convex for any positive time. Choose a positive time $t_0 \in (0, T)$, so that $H(t_0) \geq H_0 > 0$. Since $M(t_0)$ is not singular, it is smooth, and its interior region $\Omega(t_0)$ is open, so an open ball of some radius r may be placed inside $\Omega(t_0)$ and tangent to $M(t_0)$.

Set $\alpha = rH_0$. Then $M(t_1)$ is α-non-collapsed. By the main theorem of [2], $M(t)$ is α-non-collapsed with the same α for all times $t \in [t_1, T)$. It is clear the ratio $r = \frac{\alpha}{H}$ is scale invariant, so we must have that $\widetilde{M}(s)$ is also α-non-collapsed for $s \geq s_1$. Now Definition 4.1 applies. □

We want to reduce all possible singularities to spheres and cylinders which exhibit convergence in the graph sense. Thinking primarily of the main result in [43], we reduce all possible cases of tangent flows to spheres and cylinders. We begin by analyzing those two cases, relying heavily on the uniqueness of cylindrical tangent flows shown in [12].

Lemma 4.18. *Let M be as in Lemma 4.17 with the singular point instead at some $x \in \mathbb{R}^{N+1}$. Assume there is at least one tangent flow at (x, T) that is a generalized cylinder.*

Then $\lim_{s\to\infty} \widetilde{M}(s) = \widetilde{M}_\infty$ exists and is the same generalized cylinder with radius $\sqrt{m}$, where m is the dimension of the cylinder's round factor.

Convergence is smooth on compact subsets of $\mathbb{R}^3$. That is, for large s, $\widetilde{M}(s)$ can be locally described as a graph, over $\widetilde{M}_\infty$, of some function u, which is C^k-small. (See **Graph Convergence** in §2.2.) We only need $k = 2$, and will write "$\xrightarrow{C^2_c}$" for this type of convergence.

Proof. Assume, without loss of generality, that the singular point is the origin. Assume there is at least one tangent flow at $(0, T)$ that is generalized-cylindrical.

Let $s_i \nearrow \infty$. We know Lemma 4.17 provides a subsequence $\sigma = \{s_{i_j}\}$ so that $\widetilde{M}(s_{i_j})$ has a subsequence converging to some $\widetilde{M}^\sigma_\infty$ on each compact set K in the sense of graphs.

Now we want to show that $\widetilde{M}^\sigma_\infty$ is independent of subsequence. Let $\mu_i = \lambda(t_i)$, so that by (2.4),

$$M^{i_j}\left(-\frac{1}{2}\right) = \widetilde{M}(s_{i_j}) \xrightarrow[j]{C^2_c} \widetilde{M}^\sigma_\infty.$$

That means $\widetilde{M}^\sigma_\infty$ is a tangent flow. Since there is a generalized-cylindrical tangent flow by hypothesis, Theorem 0.2 of [12] says all tangent flows at $(0, T)$ are the very same generalized cylinder, including $\widetilde{M}^\sigma_\infty$.

In fact, we now have that every sequence $s_i \nearrow \infty$ has a subsequence s_{i_j} for which $\widetilde{M}^\sigma_\infty$ is the same generalized cylinder as above. Then it must be true that for every sequence of times going to ∞, $\lim_{i\to\infty} \widetilde{M}(s_i)$ exists and is the same generalized cylinder. Finally, this means that $\lim\limits_{s\to\infty} \widetilde{M}(s)$ makes sense and is a unique generalized cylinder $\widetilde{M}_\infty = \widetilde{M}^\sigma_\infty$.

For a flow to be stationary, we need

$$0 = (\partial_s \widetilde{\mathbf{F}}_\infty)^\perp = \widetilde{\mathbf{F}}^\perp_\infty - \widetilde{H}_\infty \widetilde{\nu}_\infty = \left(r - \frac{1}{r}\right) \widetilde{\nu}_\infty.$$

by (2.2). Thus, for a generalized cylinder $\widetilde{M}^\sigma_\infty = \mathbb{S}^m \times \mathbb{R}^{N-m}$, the radius must be $r = \sqrt{m}$, and $\widetilde{M}^\sigma_\infty$ must be centered around the origin.

$\square$

Lemma 4.19. *Let M be as in Lemma 4.17 with the singular point instead at some $x \in \mathbb{R}^{N+1}$. Assume there is at least one tangent flow at (x, T) that is a sphere, with radius $\sqrt{N}$.*

Then $\lim\limits_{s\to\infty} \widetilde{M}(s) = \widetilde{M}_\infty$ exists and is the same sphere.

Proof. Assume, without loss of generality, that the singular point is the origin. Assume there is some tangent flow at $(0, T)$ that is a sphere.

Then there is some sequence $\mu_i \nearrow \infty$ (assume $\mu_i > 2$) for which each $M^i(t) := M_{\mu_i}(t)$ is defined on $[-T, 0)$ whose limit flow is a sphere. More precisely, recall that $\lambda(t) = (2(T-t))^{-\frac{1}{2}}$ and $s(t) = -\frac{1}{2}\log(T-t)$ so $M(t) = \lambda^{-1}(t)\widetilde{M}(s)$. For a fixed t, choose

$$s_i = \log \mu_i - \frac{1}{2}\log(-t) \nearrow \infty.$$

Then by (2.3),

$$\begin{aligned} M^i(t) = \mu_i M(T + \mu_i^{-2}t) &= \mu_i \lambda^{-1}(T + \mu_i^{-2}t)\widetilde{M}\left(\log \mu_i - \frac{1}{2}\log(-t)\right) \\ &= \sqrt{-2t}\widetilde{M}(s_i) \end{aligned} \tag{4.2}$$

We know Lemma 4.17 provides a subsequence s_{i_j} so that $\widetilde{M}(s_{i_j})$ converges to some $\widetilde{M}^{\sigma}_{\infty}$ in the graph sense. Also, since $\widetilde{M}^{\sigma}_{\infty}$ is compact, for large j

$$\widetilde{M}_{s_{i_j}} \cap B_{2\max\{C,\sqrt{N}\}} = \widetilde{M}_{s_{i_j}},$$

and that is a single component (C is the type-I constant). Then by (4.2), we have the tangent flow $M^{\infty}(t) = \sqrt{-2t}\widetilde{M}^{\sigma}_{\infty}$.

By hypothesis, we then know $\widetilde{M}^{\sigma}_{\infty}$ is a fixed sphere. Given the C^2_c convergence and the compactness of $\widetilde{M}(s)$, that means there is some large s_1 and some $R > 0$ for which $\widetilde{M}(s_1) \cap B_R(0)$ is closed and strictly convex. Since $\widetilde{M}(s)$ is connected for all s, that means $\widetilde{M}(s_1)$ itself is closed and strictly convex. Finally, we know from Huisken's main theorem in [27] that $\widetilde{M}(s)$ remains convex after s_1 and converges to a sphere in C^2 (globally) as $s \nearrow \infty$. That means $\widetilde{M}_{\infty}$ makes sense and is a sphere. □

Definition 4.20. *Assume M has a singular point x and that $\widetilde{M}^x_\infty$ exists. We call x a spherical ((generalized) cylindrical) point if $\widetilde{M}^x_\infty$ is a sphere ((generalized) cylinder). Here the spheres and (generalized) cylinders have the radii specified in* ***Cylinders*** *from §2.2.*

Corollary 4.21. *Let M be a smoothly embedded, closed, connected, type-I, mean-convex mean curvature flow with first singular time T.*

Then the flow M

- *has at least one singular point x,*
- *all singularity blow-ups exhibit smooth convergence in the sense of graph sense.*
- *either $M(t)$ becomes convex and shrinks to the point x, or all singular points of M at time T are cylindrical.*
- *each spherical blow-up has radius $\sqrt{N}$ and each generalized-cylindrical blow up, with round factor m, has radius $\sqrt{m}$.*

Proof. Since M_0 is compact, one can place a sphere containing it. By the comparison principle, $M(t)$ must become singular before the sphere collapses. Call the first singular time T.

Without loss of generality, let x be the origin. Take a sequence of rescale times $s_i \nearrow \infty$. Again we know from Lemma 4.17 that there is a subsequence s_{i_j} for which $\widetilde{M}^{\{s_{i_j}\}}_\infty$ exists. By (2.4), that is a tangent flow. Therefore, at least one tangent flow at $(0, T)$ exists.

Since $M(t)$ is mean-convex, Theorem 1.1 of [43] says every tangent flow at $(0, T)$ is a plane, a sphere, or a cylinder. However, Corollary 8.1 in [40] precludes any planar tangent flows at the first singular time if $\{t_i\}$ is increasing. Thus the aforementioned tangent flow at $(0, T)$ is a sphere or a cylinder. Then by Lemma 4.18 and Lemma 4.19, every singular point is either spherical or cylindrical. Thus $\widetilde{M}_\infty$ exists and is either a sphere or cylinder.

Assume $\widetilde{M}_\infty$ is a sphere. Then there is some s and $R > 0$ for which $\widetilde{M}(s) \cap B_r(0)$ is closed and convex. Since $\widetilde{M}(s)$ is connected, that means $\widetilde{M}(s)$ itself is closed and convex.

Then the same is true of $M(t)$ at the corresponding time t. Therefore $M(t)$ collapses to a point, by the main theorem of [27]. Thus, the existence of spherical points and the existence of cylindrical points are mutually exclusive.

The values of the radii come from Lemmas 4.18 and 4.19.

$\square$

Chapter 5

Continuity

5.1 Introduction

5.1.1 Background

We study the solution $M(t)$ to mean curvature flow, with initial data $M(0) = M_0$, near the first singularity at time T. Let $\mathbf{F} : \mathcal{M} \times [0, T) \to \mathbb{R}^{N+1}$ be a family of smooth embeddings $\mathbf{F}(\cdot, t) = M(t)$, where $\mathcal{M}$ is a closed N-dimensional manifold. We say that $M = \{M(t)\}_{t \in [0,T)}$ is a mean curvature flow if

$$\partial_t \mathbf{F} = -H\nu, \tag{5.1}$$

where H is the scalar mean curvature, ν is the *outward* unit normal, and $-H\nu$ is the mean curvature vector.

Under this flow, a hypersurface decreases its area at each point as rapidly as possible. Because the equation is parabolic, it has a regularizing effect. Under the right conditions, curvature tends toward uniformity. One such condition, as shown by Huisken in [27], is convexity. A convex hypersurface, as it shrinks to a point, becomes asymptotically round. In fact, due to symmetry, a sphere will contract to its center, and its radius can be found via ODE.

As a parabolic equation, mean curvature flow also has a kind of comparison principle: two initially disjoint hypersurfaces cannot intersect at a later time (see §2.2). One can therefore show that any compact initial hypersurface must develop a singularity infinite time. Place a large sphere around the initial hypersurface so that it must shrink and vanish no later than than the sphere. It is also known that a singularity cannot develop without the curvature blowing up.

Shrinking to a point is not the only kind of singularity possible. A hypersurface can also develop a neck structure, for example like a barbell, with a bulb on either end. Angenent showed in [6] there exist singularities in which the neck pinches before the hypersurface can become round. He did so by placing a sphere in each bulb and a homothetically shrinking torus around the neck. This way, if the torus is small compared to the spheres, it will pinch around the neck before the spheres. Later, we use a similar construction to force a singularity by a certain time.

In order to control the types of singularities, we will distinguish between what are called type-I and type-II singularities. Type-I flows exhibit a natural bound on curvature growth

$$|A| \leq C(2(T-t))^{-\frac{1}{2}},$$

with $C > 0$, where T is the first singular time. This is the rate at which the curvature of a sphere blows up and can be found from solving the ODE obtained by assuming uniform curvature. White showed in [43] that mean-convexity restricts blow-ups at singularities to either a sphere or a generalized cylinder (as in a neck). The type-I curvature bound ensures smooth convergence in the blow-up, giving us much needed precision in describing the neck. Type-II simply means not type-I. Type-II singularities are less well-understood and are not discussed here.

We show the continuity of first singular time for two classes of flows. As a corollary, we show continuity of the limit set at time T. To do so, we need careful understanding of what can happen at a singularity under the assumptions of mean-convexity and a type-I curvature bound. See §4.1 for more background on singularities.

There are some stability results for specific hypersurfaces, like the sphere in [15], or particular neck shapes in [21, 20, 19]. However to the author's knowledge, the continuity of singular time is the first of its kind.

5.1.2 Main Results

Theorem 5.1. *Let $\overline{M}_0 \subset \mathbb{R}^{N+1}$ be a smoothly embedded, closed, mean-convex hypersurface. Let $\overline{M}(t)$ be the solution to mean curvature flow with $\overline{M}(0) = \overline{M}_0$. Assume that $\overline{M}(t)$ shrinks to a point at time $\overline{T}$.*

For every $n \in \mathbb{N}$, let $M_{n0} \subset \mathbb{R}^{N+1}$ be a smoothly embedded, closed hypersurface that can be expressed as the graph of some function f_n over $\overline{M}_0$. Finally, say $\overline{T}$ and T_n are the first singular times for $\overline{M}$ and M_n, respectively.

If $M_{n0} \to \overline{M}_0$ (i.e. $\|f_n\|_{C^2(\overline{M}_0)} \to 0$), then $T_n \to \overline{T}$.

Convergence is smooth in the sense of compact graphs over the hypersurface. By the main theorem of [27], if $\overline{M}_0$ is convex, $\overline{M}(t)$ shrinks to a point. In that case, it follows from Corollary 5.3 below that the point is continuous with respect to initial data in the same sense.

The proof of Theorem 5.1 is short and relies heavily on the inclusion monotonicity of mean-convex flow. The technique is useful in the proof of our main result Theorem 5.2, so we do the proof of Theorem 5.1 in §5.1.4, as soon as we have established general notation and definitions. The proof of Theorem 5.2 requires $N = 2$, so it is not a strict generalization of Theorem 5.1.

Why the restriction to surfaces? The technique used in proving Theorem 5.2 is inspired by the neck-pinching strategy employed by Angenent in [6] (see §5.1.3 for an overview). If $\overline{M}_0$ is a surface that does not collapse to a point under the flow, we can use current theory to predict the appropriate neck structure (i.e. the portion of $\overline{M}(t)$) near a singularity is close to a truncated cylinder, say $\mathbb{S}^1 \times [-a, a]$). In higher dimensions, other structures are possible (i.e. the portion of $\overline{M}(t)$ is close to a generalized cylinder that splits off a hyperplane, rather than a line) allowing for too many degrees of freedom in the motion of $\overline{M}(t)$ and nearby flows.

Theorem 5.2. *Let $\overline{M}_0 \subset \mathbb{R}^3$ be a smoothly embedded, closed, mean-convex surface. Let $\overline{M}(t)$ be the solution to mean curvature flow with $\overline{M}(0) = \overline{M}_0$.*

For every $n \in \mathbb{N}$, let $M_{n0} \subset \mathbb{R}^3$ be a smoothly embedded, closed surface that can be expressed as the graph of some function f_n over $\overline{M}_0$. Finally, say that $\overline{T}$ and T_n are the first singular times for $\overline{M}$ and M_n, respectively.

If $M_{n0} \to \overline{M}_0$ (i.e. $\|f_n\|_{C^2(\overline{M}_0)} \to 0$), and $\overline{M}$ is a type-I flow, then $T_n \to \overline{T}$.

A corollary to continuity of first singular time is continuity of the limit set.

Corollary 5.3. *Let $\overline{M}_0$ and the sequence $\{M_{n0}\}_n$ be as in Theorem 5.1 or Theorem 5.2. Then $M_n^* \to \bar{M}^*$ in the Hausdorff metric.*

The proof of blow-up-time continuity involves multiple cases, some of which are complex, so we include an outline of the argument in §5.1.3 below.

Theorem 5.4. *Let M_0 be a smoothly embedded, closed, mean-convex, simply connected surface. Let $M(t)$ be the solution to mean curvature flow with $M(0) = M_0$, and assume M is type-I and does not shrink to a sphere.*

Then M^* has two "bulbs" (see Definitions 5.11 and 5.12), neither of which is entirely singular.

This theorem is a combination of Corollary 4.21 and Theorem 5.22. We rephrase it here because of the structure of §5.2.

5.1.3 Idea of the Main Proof

In showing continuity of blow-up time, it is a standard application of well-posedness to conclude that $\liminf_{n\to\infty} T_n \geq \overline{T}$. Thus, for Theorems 5.1 and 5.2, it is sufficient to show that $T_n \leq \overline{T} + \varepsilon$ for large n. Assume in the following that n is large.

Due to mean-convexity, the flow, and nearby flows, move inward. A hypersurface beginning inside $\overline{M}_0$ will remain inside by a comparison principle. A small adjustment in the time parameter slides nearby flows inside $\overline{M}_0$. Thus, in the following, we can assume M_{n0} is inside $\overline{M}_0$, affording us more control over when $M_n(t)$ becomes singular.

Proving Theorem 5.1 Once we reduce to the case where M_{n0} is inside $\overline{M}_0$, the proof is nearly trivial. Since $\overline{M}(t)$ shrinks to a point, there is no escape for a hypersurface inside it. Either $M_n(t)$ becomes singular before time $\overline{T}$, or it shrinks to a point at time $\overline{T}$.

That's it for Theorem 5.1. The rest of the subsection describes the proof of Theorem 5.2.

Proving Theorem 5.2 If $\overline{M}(t)$ does not shrink to a point, we show in §4.1 that $\overline{M}(t)$ must develop cylindrical singularities. In §5.2 we show cylindrical singularities correspond to a structure with a "neck" and two "bulbs". The surface $M_n(t)$ could slip through the neck and survive in just one bulb of $\overline{M}(t)$, so we cannot count on $M_n(t)$ becoming singular just because M_{n0} is inside $\overline{M}_0$. Thus more work is required, but we can use well-posedness to force $M_n(t)$ to have a neck structure like $\overline{M}(t)$.

Nonsimply Connected Case It turns out the case when $\overline{M}_0$ is not simply connected is easier than when it is simply connected. Intuitively, any tube-like portion of $\overline{M}(t)$ will enclose a tube-like portion of $M_n(t)$, and the handle structure prevents $M_n(t)$ from wriggling away. Practically, we choose a nontrivial loop in $M_n(t)$ and a loop around the neck to create a Hopf link preserved by the flow. (See Figure 2.3a)

Simply Connected Case The strategy for the simply connected case is more intuitive, but far more technical. Inspired by Angenent's neck-pinching strategy in [6], in each bulb we place a sphere to hold it open while the neck pinches. Because mean curvature flow is well-posed, we can choose n large enough that M_n also has two bulbs held open by the spheres (see Figure 5.1). Angenent forces the neck to pinch by shrinking a donut around it (the Angenent donut). Since we do not prescribe initial data it is not clear an appropriate donut exists. However, the singular time T is given, so we have no need for the donut.

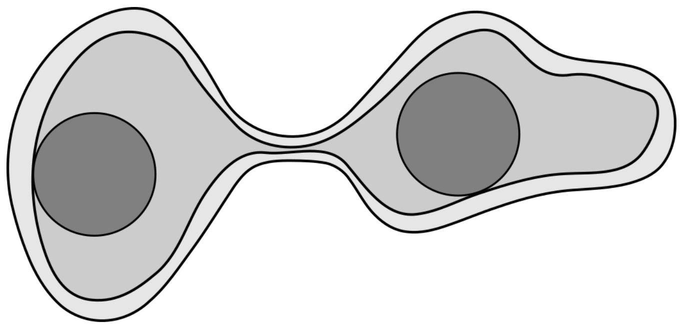

Figure 5.1: Spheres inside Ω_n with diameter much larger than that of the neck.

On the other hand, not prescribing initial data means we have no a priori knowledge of the appropriate choice of spheres. The spheres we choose must fit inside the bulbs *and* survive past the neck pinching. Thus we employ a somewhat recent tool of Andrews in [2]. Stated roughly it says that, given $\alpha > 0$ and some flow M, if at each point x of $M(t)$ a sphere of radius $r = \frac{\alpha}{H}$ can fit inside $M(t)$ tangent at x, then this condition is preserved by the flow for the same α. (See **Non-Collapsing Condition** and Figure 2.2 in §2.3) This allows us to place the spheres, as in the Angenent strategy. This is another reason we need mean-convexity.

When placing the spheres, we must choose their radius r, the time t_0 at which to place them, and n large enough that we can place them inside $M_n(t_0)$. Initially, these quantities appear circularly dependent, so we must find conditions under which one quantity can be chosen independently of the other two. The choice of n must depend on t_0 because of how we use well-posedness. The choice of t_0 must depend on r because the neck must be small compared to the spheres so that it pinches before the spheres collapse. So we have to choose $r > 0$ independent of t_0 and n. Since r is inversely proportional to the curvature, we must show the existence of points in $\mathcal{M}$ for which $H(\mathbf{F}(p,t))$ stays bounded. It is sufficient to show there is a regular (nonsingular) point in $\overline{M}^*$.

Finding a Regular Point To the author's knowledge, no result exists to guarantee there is a regular point in the limit set $\overline{M}^*$, so most of §5.2 is dedicated to finding one. Under the simply connected assumption, this begins by showing the existence of the desired neck structure with bulbs. The type-I assumption restricts the velocity at each point to prevent a bulb from collapsing into the singular point. Finally, we use some properties of the singular set to show that it cannot take up a whole bulb at time $\overline{T}$. Thus the limit of each bulb must have at least one regular point.

5.1.4 Proof of Theorem 5.1

Proof of Theorem 5.1. Let $\overline{M}$ and a sequence $\{M_n\}_n$ be as in Theorem 5.1. By well-posedness, we already have that $\liminf_{n\to\infty} T_n \geq \overline{T}$. So we need only show that $\limsup_{n\to\infty} T_n \leq \overline{T}$. For an illustration of the following, see Figure 5.2.

Let $0 < \varepsilon < \frac{T_n}{2}$. Define $\widehat{M}_n(t) = M_n(t+\varepsilon)$ so the smooth existence time interval for $\widehat{M}_n$ is $[-\varepsilon, T_n - \varepsilon)$. (We want $\varepsilon < \frac{T_n}{2}$ so the smooth existence time intervals for $\overline{M}$ and $\widehat{M}_n$ overlap by more than $\frac{\varepsilon}{2}$.)

Now $\widehat{M}_{n0} = M_n(\varepsilon)$. By the minimum principle for mean curvature, $\overline{M}(t)$ strictly is mean-convex for $t = [\frac{\varepsilon}{2}, \overline{T})$. Therefore its velocity at every point after time $t = \frac{\varepsilon}{2}$ is inward with positive speed. Thus $\overline{M}_0(\varepsilon) \subset \mathbf{\Omega}_0$, and we have the Hausdorff distance $d := d_H\left(\overline{M}(\varepsilon), \overline{M}_0\right) > 0$. By well-posedness, there is an $n_0 > 0$ so that $d_H(\overline{M}(\varepsilon), \widehat{M}_{n0}) = d_H(\overline{M}(\varepsilon), M_n(\varepsilon)) < \frac{d}{2}$ whenever $n \geq n_0$. (see Lemma 3.1). So assume $n \geq n_0$. Rearranging

$$d_H(\overline{M}(\varepsilon), \overline{M}_0) \leq d_H(\overline{M}(\varepsilon), \widehat{M}_{n0}) + d_H(\widehat{M}_{n0}, \overline{M}_0)$$

gets us

$$d_H(\widehat{M}_{n0}, \overline{M}_0) \geq d_H(\overline{M}(\varepsilon), \overline{M}_0) - d_H(\overline{M}(\varepsilon), \widehat{M}_{n0}) > d - \frac{d}{2} = \frac{d}{2} > 0.$$

Thus $\widehat{M}_{n0} \subset \mathbf{\Omega}_0$. Then, because $\overline{M}(t) \to 0$, the comparison principle tells us $\widehat{M}_n(t)$ must become singular no later than $\overline{M}(t)$ does. So we see that $T_n = \widehat{T}_n + \varepsilon \leq \overline{T} + \varepsilon$. Since that is true for any $n \geq n_0$, we have $\limsup_{n\to\infty} T_n \leq \overline{T} + \varepsilon$. Since ε was arbitrary, we are done.

□

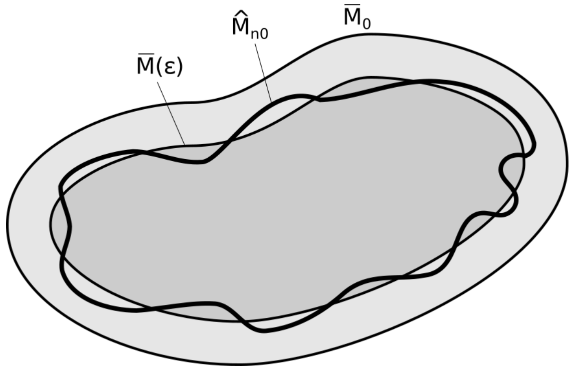

Figure 5.2: $\widehat{M}_{n0}$ is closer to $\overline{M}(\varepsilon)$ than $\overline{M}_0$, so is contained in $\mathbf{\Omega}_0$.

5.2 "Anatomy" of M

Theorem 5.2 only applies to surfaces. We can tell from the proof of Theorem 5.1 that the case when $\widetilde{M}_\infty$ is a sphere is easily resolved, since in this case $M(t)$ would shrink to a point. However, we will need much more understanding about the case when $\widetilde{M}_\infty$ is a cylinder. The key to all of our analyses is the neck structure that forms at a type-I cylindrical singularity. Nearly all the results in this section make heavy use of this structure (detailed in Lemma 5.5 and Definition 5.7), so we make the following assumptions for *this entire section*.

(i) M_0 (and therefore each $M(t)$) is a smoothly embedded, closed, connected, mean-convex *surface*.

(ii) M is type-I with constant C_0 (i.e. $|A| \leq C_0\lambda$). Although if $C_0 < 1$, set $C_0 = 1$.

(iii) M has only cylindrical singularities

(iv) The singularity in question is at the origin, and the axis of $\widetilde{M}_\infty$ is the x_2-axis

Note in §5.2.2, we add a fifth condition concerning the topology of $M(t)$.

5.2.1 Neck Formation

We need to describe very precisely what we mean by neck structure.

Lemma 5.5. *Define the solid truncated cylinder*

$$\widetilde{K} = \left\{ \xi : |\xi_2| \leq 2C_0 \text{ and } \sqrt{\xi_1^2 + \xi_3^2} \leq 2C_0 \right\}.$$

(See Figure 5.3) Let $\widetilde{\nu}_\infty$ be the outward normal on $\widetilde{M}_\infty$. The surface $\widetilde{M}_\infty$ is the unit-radius cylinder whose axis is the ξ_2-axis, and the following hold:

For every $0 < \varepsilon < \frac{1}{2}$ there is $s_\asymp > s_0$ ("s-neck") and smooth $\widetilde{f} : (\widetilde{M}_\infty \cap \widetilde{K}) \times [s_\asymp, \infty) \to \mathbb{R}$, for which the function $\widetilde{\varphi}_s : (\widetilde{M}_\infty \cap \widetilde{K}) \times [s_\asymp, \infty) \to (\widetilde{M} \cap \widetilde{K})$ defined by

$$\widetilde{\varphi}(\xi, s) := \xi + \widetilde{f}(\xi,)\widetilde{\nu}_\infty$$

is a smooth diffeomorphism, and $\|\widetilde{f}\|_{C^2} < \varepsilon$.

Remark 5.6. *Since $\|\widetilde{f}\|_{C^2} < \varepsilon < \frac{1}{2}$, $\widetilde{M} \cap \widetilde{K}$ is bounded away from the ξ_2-axis.*

Since $\|\widetilde{f}\|_{C^2} < \varepsilon < 1 \leq C_0$, $\widetilde{M}$ does not intersect the "side" of $\widetilde{K}$ ($\widetilde{K} \cap \sqrt{\xi_1^2 + \xi_3^2}$), since the radius of $\widetilde{K}$ is a least twice that of $\widetilde{M}_\infty$. Additionally $\widetilde{M}$ must intersect the "lids" of $\widetilde{K}$ ($\widetilde{K} \cap \{|\xi_2| = 2C_0\}$) transversely. Also $\widetilde{\nu}_\infty$ is parallel to the "lids" of $\widetilde{K}$, so there are no questions about the surjectivity of $\widetilde{\varphi}$.

The length of the cylinder is only needed for the proof of Lemma 5.14.

Proof. By assumption and Definition 4.20, $\widetilde{M}_\infty$ exists and is a specific cylinder with axis on the ξ_2-axis and radius $r = 1$.

Lemma 4.17 says that the convergence of $\widetilde{M}(s)$ to $\widetilde{M}_\infty$ is C^2 in $\widetilde{K}$, in the graph sense, as in §2.2. Then, for each time $s \geq s_\asymp$, we have a function

$$\widetilde{f} : (\widetilde{M}_\infty \cap \widetilde{K}) \times [s_\asymp, \infty) \to \mathbb{R}$$

for which $\widetilde{\varphi}(\xi, s) = \xi + \widetilde{f}(\xi, s)\nu_\infty$ is a smooth diffeomorphism from $(\widetilde{M}_\infty \cap \widetilde{K}) \times [s_\asymp, \infty)$ to $\widetilde{M}(s) \cap \widetilde{K}$ (due in part to Remark 5.6).

That $\|\widetilde{f}\|_{C^2} < \varepsilon$ is a direct consequence of the C^2 convergence of $\widetilde{M}(s)$ to $\widetilde{M}_\infty$ in $\widetilde{K}$. $\square$

Note: For the most part, depictions will follow:

$$\begin{array}{lcl} x_1\text{-axis} & : & \text{longitudinal (into the page)} \\ x_2\text{-axis} & : & \text{lateral} \\ x_3\text{-axis} & : & \text{vertical} \end{array}$$

And we also sometimes use x_2 as a coordinate function.

Definition 5.7 (Useful Sets and Quantities)**.** *For later use, also define the disk* $\widetilde{\mathbb{D}} = \{\xi_2 = 0\} \cap \widetilde{K}$*, orthogonal to the axis of* $\widetilde{K}$ *(and therefore the axis of* $\widetilde{M}_\infty$*).*

We write K *and* $\mathbb{D}$*, without tildes to denote their nonrescaled versions. That is* $K(t) = \lambda^{-1}(t)\widetilde{K}$ *and* $\mathbb{D}(t) = \lambda^{-1}(t)\widetilde{\mathbb{D}}$*. Then we can write* $M_\asymp(t) = M(t) \cap K(t)$*, and* $\Omega_\asymp(t) = Int(\Omega(t) \cap K(t))$ *("*M*-neck" and "*Ω*-neck").*

Throughout this paper, we refer to $t_\asymp$ *("t-neck"), corresponding to* $s_\asymp$*. After time* $t_\asymp$ *(*$s_\asymp$*), we say* $M(t)$ *(*$\widetilde{M}(s)$*) "has a neck" for* $t \in [t_\asymp, T)$ *(*$s_\asymp \in [s_\asymp, \infty)$*). Any mention of* $t_\asymp$ *(*$s_\asymp$*) hereafter implies all the structures given in Lemma 5.5 and Definition 5.7 are present in* $M(t)$ *(*$\widetilde{M}(s)$*). (See Figure 5.4)*

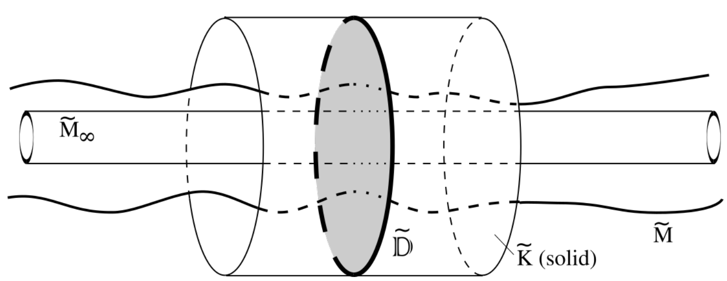

Figure 5.3: $\widetilde{K}$ and $\widetilde{\mathbb{D}}$.

Remark 5.8. *Note that* $K(t)$ *is shrinking homothetically, so* $M_\asymp(t) = M(t) \cap K(t)$ *is not the same as the image of* $M_\asymp(t_\asymp)$ *under mean curvature flow. It is certainly not true that if* $\mathbf{F}(p,t) \in K(t)$ *for some* t*, that* $\mathbf{F}(p,t)$ *must stay in* $K(t)$ *for later times.*

The effects of the subscript in $M_\asymp$ *and* $t_\asymp$ *are* not *analogous.*

5.2.2 Existence of Bulbs

The set $M(t) \setminus M_{\asymp}(t)$ can have one or two components, and the proof of the main theorem is different in each case. Let us first make sure these cases are well-defined in the sense that the number of components cannot change in time.

Lemma 5.9. *For a set $X \in \mathbb{R}^{N+1}$, let $\#X$ be the number of connected components of X. We have the following three facts about the topology of $M(t)$ after $t_{\asymp}$:*

(a) $M(t_{\asymp}) \setminus M_{\asymp}(t_{\asymp})$ is 1 or 2 path-connected components.

(b) That number is preserved in the sense that for every $t \in [t_{\asymp}, T)$,

$$\#[M(t) \setminus M_{\asymp}(t)] = \#[M(t_{\asymp}) \setminus M_{\asymp}(t_{\asymp})].$$

(c) Similarly for the interior region, for every $t \in [t_{\asymp}, T)$,

$$\#[\Omega(t) \setminus Cl(\Omega_{\asymp}(t))] = \#[M(t) \setminus M_{\asymp}(t)].$$

Proof. (a) Because $M(t_{\asymp})$ is connected, the closure of each component of $M(t_{\asymp}) \setminus M_{\asymp}(t_{\asymp})$ intersects the closure of $M_{\asymp}(t_{\asymp})$. We also know from Lemma 5.5 that $\#[\partial M_{\asymp}(t_{\asymp})] = 2$. Thus any point in $M(t_{\asymp}) \setminus M_{\asymp}(t_{\asymp})$ is path-connected to at least one of the two components of $\partial M_{\asymp}(t_{\asymp})$, so $M(t_{\asymp}) \setminus M_{\asymp}(t_{\asymp})$ has at most two components. Clearly by Lemma 5.5, $M(t_{\asymp}) \setminus M_{\asymp}(t_{\asymp})$ is not empty. Therefore the first item is proven.

(b) If, for any time $t \in [t_{\asymp}, T)$, $M(t) \setminus M_{\asymp}(t)$ is only one component, then a nontrivial loop passing through $M_{\asymp}(t)$ nontrivially can be constructed:

First draw a curve in $M(t) \setminus M_{\asymp}(t)$ connecting the two components of its boundary (which is also $\partial M_{\asymp}(t)$), then closing the loop with a curve in $M_{\asymp}(t)$ (see Figure 2.3a). This loop cannot be removed from $M_{\asymp}(t)$ via homotopy, and it forms a Hopf link with $\mathbb{D}(t)$.

Now we observe that for, $t_1, t_2 \in [0, T)$, $F : \mathcal{M} \times [t_1, t_2]$ is a homotopy. Furthermore, since $\mathbb{D}(t)$ shrinks homothetically with time, it also undergoes homotopy. Well, the Hopf link is homotopy invariant, so if for *any* $t \in [t_\asymp, T)$ there is a loop in $M(t)$ passing nontrivially through $M_\asymp(t)$, then the same is true for *all* $t \in [t_\asymp, T)$. This of course is only possible if $M(t) \setminus M_\asymp(t)$ is only one component.

(c) Each component of $\partial M_\asymp(t)$ is the boundary of some set in $K(t)$ homeomorphic to disk. Thus a component of $M(t) \setminus M_\asymp(t)$ can be closed off by union with one or both of these disks. The resulting set is embedded and connected, so it has a connected interior. $\square$

Remark 5.10. *By the proof of Lemma 5.9 above, $M_{=\!\circ}(t)$ does not intersect $Cl(\Omega_{\circ\!=}(t))$ and vice versa. This fact is important to keep in mind later in this section when we assume that the inward motion does not allow points of $M_{=\!\circ}(t)$ to move into $\Omega_{\circ\!=}(t)$. Likewise, points of $M_{=\!\circ}(t)$ never enter any part of $\Omega_{\circ\!=}(t_\asymp)$.*

Now we give some notation regarding the case of two components.

Definition 5.11. *For $t \in [t_\asymp, T)$, assume $M(t) \setminus M_\asymp(t)$ has exactly two components. Call them $M_{=\!\circ}(t)$ and $M_{\circ\!=}(t)$, chosen so that $x_2(Cl(M_{\circ\!=}(t)) \cap K(t)) < 0$ and $x_2(Cl(M_{=\!\circ}(t)) \cap K(t)) > 0$. Similarly define $\Omega_{\circ\!=}(t)$ and $\Omega_{=\!\circ}(t)$ to be the two components of $\Omega(t) \setminus Cl(\Omega_\asymp(t))$ with $x_2(\Omega_{\circ\!=}(t) \cap K(t)) < 0$ and $x_2(\Omega_{=\!\circ}(t) \cap K(t)) > 0$.*

We may use "bulbs" to refer to $M_{\circ\!=}(t)$ and $M_{=\!\circ}(t)$ or *$\Omega_{\circ\!=}(t)$ and $\Omega_{=\!\circ}(t)$. It should be clear from context whether we mean the surface or its interior region.*

5.2.3 Preservation of Bulbs

In the proof of Theorem 5.2, the case where $M(t_\asymp) \setminus M_\asymp(t_\asymp)$ is one connected component is simpler than when it is two, due to the the Hopf link construction in §2.3. So we deal with that case directly in §5.3. Thus in addition to conditions (i)-(iv), for the *rest of this section* we assume:

(v) $M(t_\asymp) \setminus M_\asymp(t_\asymp)$ has exactly two components.

In the case that M without the neck is two components, we need to place spheres inside the bulbs, as in Figure 5.1. In order to place the spheres as planned, we use the non-collapsing condition. This necessarily means we need regularity somewhere as $t \to T$. To even address this, we need to ensure $M(t) \setminus M_{\asymp}(t)$ does not vanish as $t \to T$ in this subsection. Then §5.2.4 is devoted to demonstrating the necessary regularity.

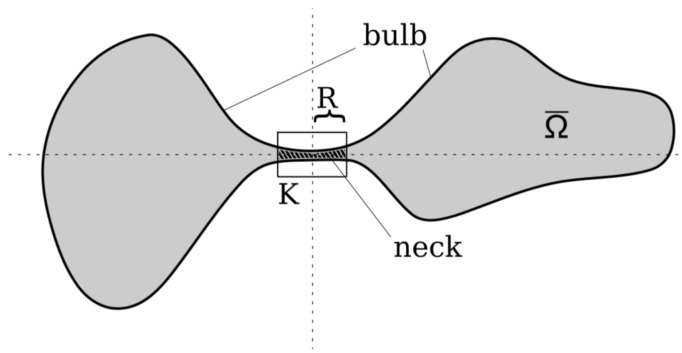

Figure 5.4: Neck in $K(t)$.

The goal is to find, for each bulb, a $p \in \mathcal{M}$ for which $H(\mathbf{F}(p,t))$ stays bounded. However, in order to do that, we need to know that the bulbs do not vanish altogether by shrinking into the origin. Even though each bulb has points outside the neck throughout the flow, either bulb could shrink into the origin at a rate slower than the neck does.

Once we know the existence of limit bulbs outside the origin, then we can address their regularity. We defined the limit bulbs in a similar way to M^*.

Definition 5.12. *Define $M^*_{=\!\circ}(t)$ to be the set of all points $x \in \mathbb{R}^3$ such that there is a sequence $(x_i, t_i) \in \mathbb{R}^3 \times [0, T)$ with $t_i \nearrow T$, $x_i \in M_{=\!\circ}(t_i)$ and, $x_i \to x$.*

Define $M^*_{\circ\!=}$ similarly.

Remark 5.13. *Note $M^*_{=\!\circ}$ will always include the origin.*

Lemma 5.14. *Neither bulb shrinks to the origin. That is, there is a nonzero $x \in M^*_{=\!\circ}$. Likewise for $M^*_{\circ\!=}$*

After reading the proof of Proposition 5.25, one may look back here at Lemma 5.14 and wonder why it doesn't solve the whole problem by preventing M_n from slipping through the neck. We remind the reader that M_n is not necessarily type-I, so we cannot apply the lemma. We have no bound on the velocity of points in M_n.

Proof. Without loss of generality, we only consider $M_{\multimap}$. We do the argument in terms of p^* then convert it to a sequence of $x_i \in M_{\multimap}$ to relate it to $M^*_{\multimap}$.

Let $p \in \mathcal{M}$ and write

$$x(t) := \mathbf{F}(p, t_{\asymp}) \in M_{\multimap}(t_{\asymp}).$$

By definition of $K(t)$, we have $|x(t)| > 2C_0\lambda^{-1}(t_{\asymp})$. By Lemma 3.2,

$$|p^* - x(t)| \leq C_0\lambda^{-1}(t_{\asymp}).$$

Therefore $p^* \geq C_0\lambda^{-1}(t_{\asymp}) > 0$. Now take any sequence $t_i \nearrow T$ and consider $x_i = \mathbf{F}(p, t_i)$. Then $x_i \in M_{\multimap}(t_i)$ and $x_i \to p^* \neq 0$ and we are done. $\square$

Now that we've established that $M^*_{\multimap}$ and $M^*_{\multimapinv}$ have some points left to work with, we set out to make sure each limit bulb has at least one regular point. We do so in the next subsection.

5.2.4 Topology of the Limit Bulbs

Recall conditions (i)-(v) are still assumed.

In this subsection, we show that neither bulb can have an entirely singular limit set. Thence we conclude the preimage of each bulb has a point p for which $H(\mathbf{F}(p, t))$ stays bounded. That will eventually allow us to find a regular point at which to place the spheres in the main proof.

We only need to find one regular point on each limit bulb, and the intuition for the argument is as follows. If the whole bulb becomes singular, we can consider its "farthest" point from the origin. Then by assumption that point is a cylindrical singular point. However, that would be strange since there should then be more $M^*_{\multimap}$ on the other side of the point, so it would not be the farthest. That is a contradiction. Thus the next few steps are to show that M^* is path-connected and that the origin separates it into the two limit bulbs.

The argument clearly assumes that $M^*_{\multimap}$ does, in fact, have a farthest point. We therefore start with one lemma showing intrinsic distance can be defined in $M^*_{\multimap}$. Namely that it is path-connected by finite-length paths.

Lemma 5.15. *The limit set M^* is Lipschitz path-connected.*

Proof. Since $\mathcal{M}$ is, by assumption, compact and path-connected, this follows immediately from the Lipschitzness of $\mathbf{F}^*$ shown in Lemma 3.4. $\square$

The preceding Lemma 5.15 ensures that even if $M^*_{\multimap}$ (or $M^*_{\multimapinv}$) is entirely singular, the intrinsic distance between any two points in $M^*_{\multimapinv}$ (or $M^*_{\multimapinv}$) is finite. However, this is not useful if $M^*_{\multimap} \setminus \{0\}$ somehow intersects $M^*_{\multimapinv} \setminus \{0\}$, since we could not identify the "farthest" point in each bulb (ultimately we want that the singular point disconnects M^*).

Before showing the singular point disconnects M^*, it is geometrically easier to show that $\mathbb{D}$ disconnects M^*, then to show that M^* can only intersect $\mathbb{D}$ at 0. This makes intuitive sense since the neck is shrinking to a point. We must then check that the bulbs to not fold back across $\mathbb{D}(t_{\asymp})$ as in Figure 5.5.

We show this in the following technical result Lemma 5.17. Really the only tool used is the monotonicity of the flow. However, $\Omega_{\multimapinv}(t)$ and $\Omega_{\multimap}(t)$ are not monotonic, so we modify them with the following definitions.

Definition 5.16. *Define $\Omega_R(t) := \Omega_{\multimap}(t) \cup (\Omega_{\asymp} \cap \{x_2 \geq 0\})$.*

Do likewise for Ω_L, but with $x_2 \leq 0$.

Lemma 5.17. *Let $m = \operatorname{dist}(M_{\asymp}(t_{\asymp}), x_2\text{-axis})$. For every $\varepsilon \in (0, m)$, and*

$$x \in \bigcup_{t \in [t_{\asymp}, T)} M_{\multimap}(t),$$

we have that if $\operatorname{dist}(x, x_2\text{-axis}) > \varepsilon$, then $\operatorname{dist}(x, \mathbb{D}(t_{\asymp})) > \varepsilon$.

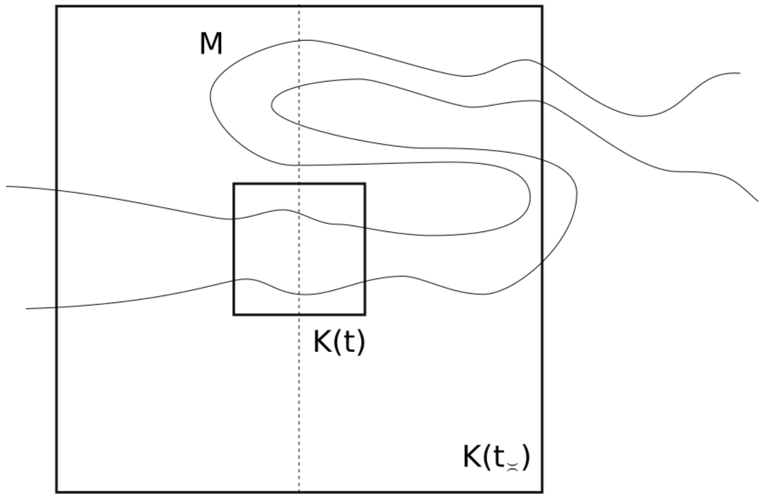

Figure 5.5: Right bulb folding back toward the neck.

Note the cross section of $K(t)$ is a square by definition of $K(t)$.

The idea here is that $M_{\asymp}$ sweeps out most of $\Omega_{\asymp}(t_{\asymp})$ due to the foliation of $\Omega(t_{\asymp})$, but not all because the neck and bulbs are defined in terms of K, not just the flow. Since $K(t)$ is shrinking, we expect a rough cone shape in $\Omega_{\asymp}(t_{\asymp})$ that is actually swept by $M_{\subset}$ and $M_{\supset}$. Though the cone may be difficult to describe, in the statement of the lemma we model its effect by restricting our attention to a roughly rectangular shape $\mathcal{A}$ (in cross section) above the origin, which is thinner when taller (compare Figure 5.6 with Figure 5.7), that only allows points in the bulb to approach $\mathbb{D}(t_{\asymp})$ if they approach the origin.

Proof. First, recall by Lemma 5.5, that $\widetilde{M}_{\asymp}(s)$ is described by $\widetilde{f}(\xi_2, \theta, s)$ on $[-2C_0, 2C_0] \times 2C_0\mathbb{S}^1 \times [s_{\asymp}, \infty)$. In the nonrescaled setting, $M_{\asymp}(t)$ can be described by $f(x_2, \theta, t)$ on $[-2C_0\lambda^{-1}(t'), 2C_0\lambda^{-1}(t')] \times \mathbb{S}^1 \times [t_{\asymp}, t')$ for any fixed $t' \in (t_{\asymp}, T)$. The domain must be adjusted because $K(t)$ is shrinking, so the available x_2 interval is also shrinking. Therefore we choose the smallest x_2 interval so that it works for all $t \in [t_{\asymp}, t')$. The radius of $\mathbb{S}^1$ is fixed at 1 just so that the domain is not changing, but this only has the effect of "vertically" shifting f so that it is not quite just $\widetilde{f}$ unscaled. Next we need to choose t' large enough so that $M_{\asymp}(t')$ is within ε of the x_2-axis so we can make use of the region $M_{\asymp}$ sweeps out.

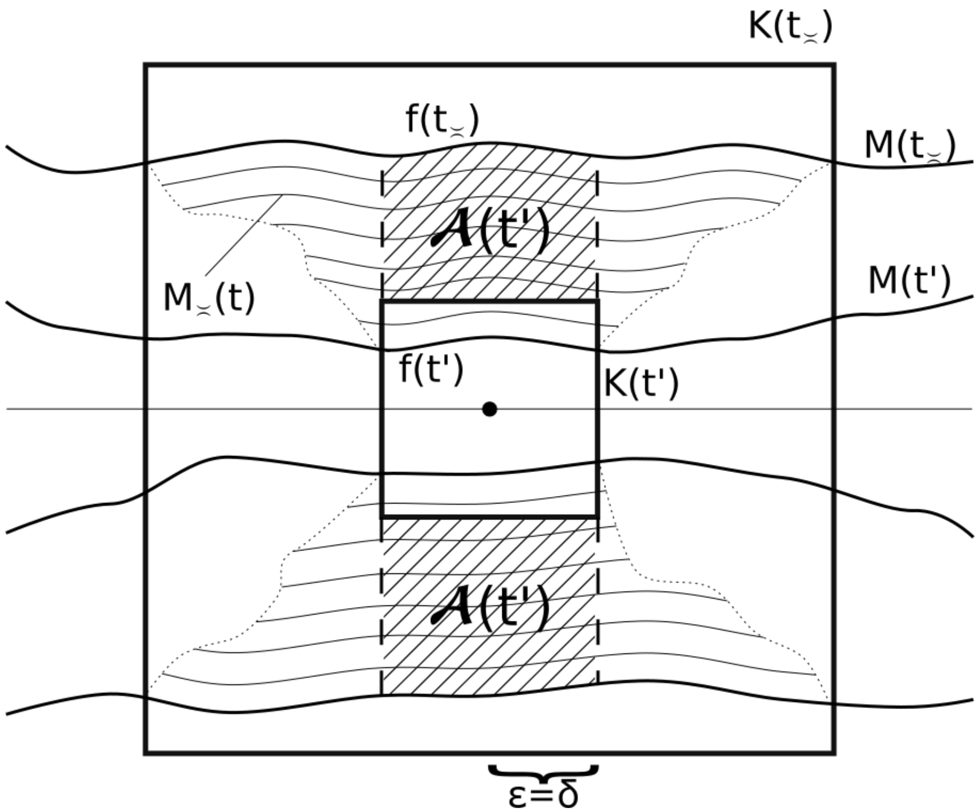

Figure 5.6: The set $\mathcal{A}(t')$ is swept out by the neck and cannot contain points from bulbs. The cross section of K is square.

We observe that m from the statement of the lemma is $m = \min f(\cdot, \cdot, t_{\asymp})$. Recall the radius of $K(t)$ is $2C_0\lambda^{-1}(t)$. Now let $0 < \varepsilon < m$ and choose $t' \in (t_{\asymp}, T)$ so that $2C_0\lambda^{-1}(t') = \varepsilon$. Note that the diameter and length of $K(t')$ are both 2ε (that is, the cross section of $K(t)$ is a square). We then see that points in $M_{\asymp}(t_{\asymp}) \cap \{|x_2| \leq \varepsilon\}$ are described by $(x_2, \theta, f(x_2, \theta, t_{\asymp}))$, where $f(\cdot, \cdot, t_{\asymp}) \geq m$, and points in $M_{\asymp}(t')$ are described by $(x_2, \theta, f(x_2, \theta, t'))$, where $f(\cdot, \cdot, t') < \varepsilon < m$.

Define

$$\mathcal{A}(t) := \{|x_2| < \varepsilon\} \cap \left\{ \varepsilon < \sqrt{x_1^2 + x_3^2} \leq f(x_2, \theta, t_{\asymp}) \right\}$$

so $\mathcal{A}(t) \subset W_{\asymp}(t_{\asymp})$. Since f is continuous in t, and $f(\cdot, \cdot, t') < \varepsilon$, every point in $\mathcal{A}(t')$ is in $M_{\asymp}(t)$ for some $t \in (t_{\asymp}, t')$. That is, $\mathcal{A}(t)$ is swept out by $M_{\asymp}(t) \cap \{|x_2| \leq \varepsilon\}$ as t goes from

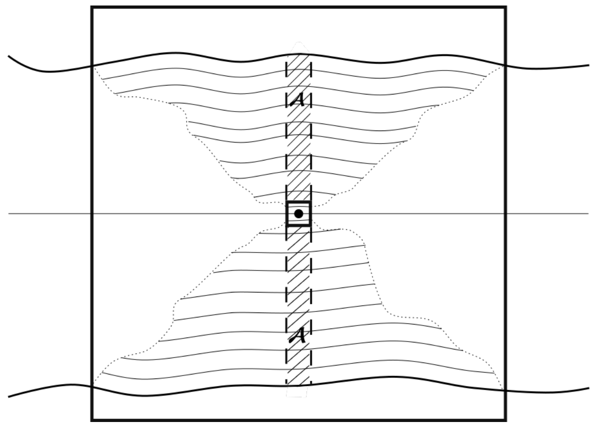

Figure 5.7: For a later t', $K(t')$ is smaller and $\mathcal{A}(t')$ is thinner. Their union is still of positive width around $\mathbb{D}$.

$t_{\asymp}$ to T. (Essentially we are making the use of the foliation quantitative.)

Thus there is no $t \in (t_{\asymp}, T)$ such that $M_{\Rightarrow}(t)$ and $\mathcal{A}(t')$ intersect. Now let

$$x \in \bigcup_{t \in [t_{\asymp}, T)} M_{\Rightarrow}(t),$$

and suppose $\operatorname{dist}(x, x_2\text{-axis}) > \varepsilon$. But then $x \in \mathcal{A}(t')$, and we have a contradiction. $\square$

Lemma 5.17 is only used to prove Theorem 5.18 below.

Theorem 5.18. *The set $M^* \setminus \{0\}$ consists of two disjoint sets $M^*_{\Leftarrow} \setminus \{0\}$ and $M^*_{\Rightarrow} \setminus \{0\}$, since Ω_L and Ω_R are disjoint. Furthermore, $M^*_{\Leftarrow}$ and $M^*_{\Rightarrow}$ are each path-connected.*

Proof. The disjointedness follows from showing that $M^*_{\Rightarrow} \setminus \{0\} = M^* \cap \Omega_R(t_{\asymp})$, and likewise for the left. That is the first two steps of the proof. The last two show the two conclusions.

Step One: $M^*_{\Rightarrow} \setminus \{0\} \subseteq M^* \cap \Omega_R(t_{\asymp})$ **(and similar for the left)**

By the Jordan-Brouwer separation theorem, $\Omega(t_{\asymp})$ is a path-connected set. Lemma 5.5 tells us that $\Omega_{\asymp}(t_{\asymp})$ is disconnected into two path-connected components on either side of the plane $\{x_2 = 0\}$. Since $\Omega_{\Leftarrow}(t)$ and $\Omega_{\Rightarrow}(t)$ are also path-connected by Jordan-Brouwer, and recalling that $\Omega(t) = \Omega_{\multimapinv}(t) \cup \Omega_{\asymp}(t) \cup \Omega_{\multimap}(t)$, we have that $\Omega_L(t_{\asymp})$ and $\Omega_R(t_{\asymp})$ are indeed path-connected.

Let $x \in M^*_{\multimap} \setminus \{0\}$. Then there is some sequence $(x_i, t_i) \in \mathbb{R}^{N+1} \times [0, T)$ such that $t_i \nearrow T$, $x_i \in M_{\multimap}(t_i)$, and $x_i \to x$. No point of $M_{\multimap}$ can be in $\Omega_L(t_{\asymp})$ for the following reasons. Since $M_{\multimap}(t)$ is path-connected and, by definition and Lemma 5.5, has points in $\Omega_R(t_{\asymp})$, if it had any point in $\Omega_L(t_{\asymp})$, there would be a path in $M_{\multimap}(t)$ connecting the point in $\Omega_L(t_{\asymp})$ to a point in $\Omega_R(t_{\asymp})$. Therefore there would be a point in $M_{\multimap}(t)$ along the path that is also in $\mathbb{D}(t_{\asymp})$, but $\mathbb{D}(t_{\asymp}) \subset \mathcal{A}(t) \cup K(t)$ for all $t \in [t_{\asymp}, T)$! Furthermore, by definition $M_{\multimap}(t)$ has no points in $K(t)$, and Lemma 5.17 tells us that $M_{\multimap}(t)$ has no points in $\mathcal{A}(t)$, so it has no points in $\mathbb{D}(t_{\asymp})$.

So each x_i is in $\Omega_R(t_{\asymp})$, but what about their limit x? Suppose $x \in \mathbb{D}(t_{\asymp}) \setminus \{0\}$. Then $|x| = \operatorname{dist}(x, x_2\text{-axis}) > 0$. Apply Lemma 5.17 with $\varepsilon = \frac{|x|}{2}$, so that for enough large i, $\operatorname{dist}(x_i, \mathbb{D}(t_{\asymp})) > \frac{\varepsilon}{2}$. We have a contradiction, so if $x \in \mathbb{D}(t_{\asymp})$, then $x = 0$.

Of course the same argument works to show $M^*_{\multimapinv} \setminus \{0\} \subseteq M^* \cap \Omega_{L(t_{\asymp})}$.

Step Two: $M^* \cap \Omega_R(t_{\asymp}) \subseteq M^*_{\multimap} \setminus \{0\}$

Let $x \in M^* \cap \Omega_R(t_{\asymp})$. Then $x \in M^*$. Recalling $0 \in M^*_{\multimap}$, since $M_{\asymp}(t) \to 0$, $M^* = M^*_{\multimap} \cup M^*_{\multimapinv}$. Since $x \neq 0$ by assumption, if x were in $M^*_{\multimapinv}$ we know from the previous argument that $x \in \Omega_L(t_{\asymp})$, which is a contradiction since $x \in \Omega_R(t_{\asymp})$. Therefore since $x \neq 0$, $x \in M^*_{\multimap} \setminus \{0\}$.

Step Three: $M^*_{\multimapinv} \setminus \{0\}$ and $M^*_{\multimap} \setminus \{0\}$ are disjoint

Since $M^*_{\multimapinv} \setminus \{0\} \subseteq \Omega_L(t_{\asymp})$ and $M^*_{\multimap} \setminus \{0\} \subseteq \Omega_R(t_{\asymp})$, and $\Omega_L(t_{\asymp})$ and $\Omega_R(t_{\asymp})$ are disjoint, $M^*_{\multimapinv} \setminus \{0\}$ and $M^*_{\multimap} \setminus \{0\}$ are disjoint. Furthermore, since $M^* = M^*_{\multimapinv} \cup M^*_{\multimap}$, we know from the preceding that $M^* \cap \mathbb{D}(t_{\asymp}) = \{0\}$. Finally, recall M^* is connected by Lemma 5.15. Thus

$M^* \setminus \{0\}$ is the union of the two disjoint sets $M^*_{\multimapinv} \setminus \{0\}$ and $M^*_{\multimap} \setminus \{0\}$.

Step Four: $M^*_{\multimapinv}$ and $M^*_{\multimap}$ are each path-connected

Recall from Lemma 5.15 that M^* is path-connected. Consider any two points in $M^*_{\multimap}$ and a path in M^* connecting them. If the path never enters $M^*_{\multimapinv}$, then we are done. If the path does enter $M^*_{\multimapinv}$, then we know from **Step One** that the path enters $\Omega_L(t_{\asymp})$.

We can see from **Step Three** that the only way from $M^*_{\multimap}$ to $M^*_{\multimapinv}$ in M^* is through $\mathbb{D}(t_{\asymp})$, and therefore 0. There must be a first and last time the curve passes through 0, so choose a new curve by only keeping the parts before the first time and after the last time. We then have a continuous curve in $M^*_{\multimap}$ (which includes 0) that connects the original two points.

$\square$

Corollary 5.19. *M^* is exactly two components.*

Remark 5.20. *This result is not needed in this work, but may be of independent interest.*

Proof. We take from the previous Theorem 5.18 that

$$M^* = (M^*_{\multimapinv} \setminus \{0\}) \cup \{0\} \cup (M^*_{\multimap} \setminus \{0\}).$$

So we need only show that $M^*_{\multimap}$ is connected.

Since $F(\cdot, t)$ is a homeomorphism for $t \in [t_{\asymp}, T)$, we know $F^{-1}(M_{\multimap}(t), t)$ is path-connected. We note that by Lemma 5.14, the set $F^{-1}(M_{\multimap}(t), t)$ is increasing in t, since once the image of a point $F(p, t)$ is not in $K(t)$, it can never reenter K. We can then come to the conclusion that

$$\bigcup_{t \in [t_{\asymp}, T)} F^{-1}(M_{\multimap}(t), t)$$

is path connected.

Claim:

$$F^* \left(\bigcup_{t \in [t_{\asymp}, T)} F^{-1}(M_{\multimap}(t), t) \right) = M^*_{\multimap} \setminus \{0\}.$$

Since $F^* \circ F^{-1}(\cdot, t)$ is continuous, we will be done once the claim is shown.

Proof of Claim:

• Let

$$p \in \bigcup_{t \in [t_{\asymp}, T)} F^{-1}(M_{\multimap}(t), t).$$

By Lemma 5.14, $p^* \in M^*_{\multimap}$. By Lemma 3.2 $p^* \neq 0$. ✓

• Let $x \in M^*_{\multimap} \setminus \{0\}$. Then there exists $p \in \mathcal{M}$ such that $p^* = x$.

If $F(p, t) \in M_{\multimapinv}(t)$ for *any* $t \in [t_{\asymp}, T)$, then by Lemma 5.14, $p^* \in M^*_{\multimapinv}$. But this is a contradiction with the assumption since $p^* = x \neq 0$.

If $F(p, t) \in M_{\asymp}(t)$ for *all* t after some $t' \in [t_{\asymp}, T)$, then $p^* = 0$, which again is a contradiction. ✓

Now we have proven the claim. □

Remark 5.21. *Although not needed directly for the proof, it may help the reader to note*

$$\mathcal{M} = (F^*)^{-1}(M^*_{\multimapinv} \setminus \{0\}) \cup (F^*)^{-1}(\{0\}) \cup (F^*)^{-1}(M^*_{\multimap} \setminus \{0\})$$

$$\textit{and } (F^*)^{-1}(M^*_{\multimap} \setminus \{0\}) = (F^*)^{-1}(M^*_{\multimap}) \setminus (F^*)^{-1}(\{0\}).$$

5.2.5 Regular Point in the Limit Bulb

Theorem 5.22. *Neither $M^*_{\multimap}$ nor $M^*_{\multimapinv}$ is entirely singular.*

Proof. Suppose that $M^*_{\multimap}$ is entirely singular. We know from Lemma 5.15, that $M^*_{\multimap}$ is path-connected. By Lemma 5.14, it is not a singleton. Finally, we observed in Remark 5.13 that $M^*_{\multimap}$ must also contain the origin.

We know from Lemma 5.15 and Theorem 5.18, that there is a finite-length path connecting the origin to each point in $M^*_{\multimap}$. In particular, there is a path in $M^*_{\multimap}$ from the origin to the intrinsically farthest point in $M^*_{\multimap}$, which exists since $M^*_{\multimap}$ is compact. That endpoint, call it x, attains the maximum intrinsic distance from 0 in $M^*_{\multimap}$.

By the opening supposition, x is a singular point, which we have also assumed to be cylindrical. This leads us to a contradiction, since by Lemma 5.14, x has its own mutually disjoint left and right limit bulbs. Any path from the origin to a point in the right bulb of x must pass through x, and is thus longer than the intrinsic distance from 0 to x. Thus x cannot be the farthest point and we have a contradiction, and $M^*_{\multimap}$ is not entirely singular. The same argument applies to $M^*_{\multimapinv}$. $\square$

Corollary 5.23. *There is at least one point $x \in M^*_{\multimap}$ with a point $p \in \mathcal{M}$ such that $p^* = x$ and $H(\mathbf{F}(p,t))$ stays bounded as $t \nearrow T$. The same is true of $M^*_{\multimapinv}$.*

Proof. Let $x \in M^*$ be nonsingular. By Lemma 3.4 there is a $p \in \mathcal{M}$ such that $p^* = x$. Then by the definition of singular point given in §2.2, there cannot be a sequence t_i so that $H\left(\mathbf{F}(p,t_i)\right) \to \infty$. Therefore $H\left(\mathbf{F}(p,t)\right)$ is bounded.

$\square$

5.3 Continuity of Singular Time

Here we set out to prove our main result.

The case where $M_{n0} \subset \mathbf{\Omega}_0$ offers much more control over the behavior of M_n in terms of the behavior of $\overline{M}$. Recall in this case, because of well-posedness, we only need to show that $T_n \leq \overline{T}$ for large n. Once we ensure the conclusion of Theorem 5.2 holds in that case, we use an argument like that of Theorem 5.1 to finish the proof of Theorem 5.2, addressing the limit of T_n more generally.

We break the work into two propositions, corresponding to the cases when $\overline{M}(t) \setminus \overline{M}_{\asymp}(t)$ is one or two components (recall this condition is preserved in time by Lemma 5.9). The two component case is easier, since we can use an argument similar to that in Lemma 5.9, so we deal with that case first.

Proposition 5.24. *Let $\overline{M}_0$ be a smoothly embedded, closed surface. Let M_{n0} be a sequence of smoothly embedded, closed surfaces such that $M_{n0} \to \overline{M}_0$ and $M_{n0} \subset \mathbf{\Omega}_0$. Assume $\overline{M}(t_{\asymp}) \setminus \overline{M}_{\asymp}(t_{\asymp})$ is* one component*, and $\overline{M}$ is type-I.*

Then there is an $n_0 > 0$ so that $T_n \leq \overline{T}$ whenever $n > n_0$.

(For related illustrations, see Figures 5.3 and 5.8.)

Proof. If $\overline{M}(t)$ is not connected, simply replace it by any of its connected components that develops a singularity at time $t = \overline{T}$. Also replace $M_n(t)$ by its connected component that is a graph over the chosen $\overline{M}(t)$. This does not change the first singular time.

We know from the proof of Lemma 5.9 that $\overline{M}(t_{\asymp})$ is not simply connected. Since $\overline{\mathbf{F}}(\cdot, t)$ is an embedding for each $t \in [0, T)$, $\overline{M}(t)$ is not simply connected for any time up to T. Therefore $\overline{M}(t)$ is never convex before T, so by Corollary 4.21 all singularities of $\overline{M}$ at time T are cylindrical. Assume, without loss of generality, that the cylindrical singularity in question is at the origin. Further assume the axis of the cylinder is the x_2-axis, and $t \in [t_{\asymp}, \overline{T})$, so we can make use of $\mathbb{D}(t)$, which we can do by Lemma 5.5.

Recall that mean curvature flow is well-posed and $\overline{M}(t)$ is compact for $t \in [0, \overline{T})$. Then there is n_0 after which $M_n(t_{\asymp})$ is a graph over $\overline{M}(t_{\asymp})$, so $M_n(t_{\asymp}) \cong \overline{M}(t_{\asymp})$. Assume $n > n_0$.

As in the proof of Lemma 5.9, we can choose a closed curve $\gamma_{t_{\asymp}} \subset \overline{M}(t_{\asymp})$ that is not contractible to a point within $\overline{M}(t_{\asymp})$ and passes through $\mathbb{D}(t_{\asymp})$ exactly once. Then we can choose a closed curve $\gamma_{n,t_{\asymp}} \subset M_n(t_{\asymp})$ that is not contractible to a point within $M_n(t_{\asymp})$ and also passes through $\mathbb{D}(t_{\asymp})$ exactly once. Thus $\gamma_{n,t_{\asymp}}$ forms a Hopf link with $\partial\mathbb{D}(t_{\asymp})$.

We would like to subject $\gamma_n(t) \subset M_n(t)$ to the flow of M_n, with initial condition $\gamma_{n,t_\asymp}$ at time $t_\asymp$. To that end, for $t \in (t_\asymp, \overline{T})$, define

$$\gamma_n(t) := \mathbf{F}_n(\mathbf{F}_n^{-1}(\gamma_{n,t_\asymp}, t_\asymp), t).$$

By Lemma 5.5, $\overline{M}(t) \cap \mathbb{D}(t)$ remains a closed curve. Furthermore, since the flow preserves the condition $M_n(t) \subset \mathbf{\Omega}(t)$, we have that $M_n(t) \cap \mathbb{D}(t)$ is bounded away from the circle $\partial\mathbb{D}(t)$ for $t \in [t_\asymp, \overline{T})$, so the point $\gamma_n(t) \cap \mathbb{D}(t)$ is also bounded away from $\partial\mathbb{D}(t)$ (uniformly in t, although that is not essential). Since the curve $\partial\mathbb{D}(t)$ shrinks homothetically, and mean curvature flow is continuous, both $\partial\mathbb{D}(t)$ and $\gamma_n(t)$ undergo homotopy without intersecting. Therefore, the Hopf link formed by $\gamma_n(t_\asymp)$ and $\partial\mathbb{D}(t_\asymp)$ is preserved until time $\overline{T}$.

In the same vein, since $M_n(t) \subset \mathbf{\Omega}(t)$, each component of $M_n(t) \cap \mathbb{D}(t)$ is a closed curve. Also, because of the Hopf link, for every $t \in [t_\asymp, \overline{T})$ there is at least one point in $\mathbb{D}(t)$ at which $\gamma_n(t)$ intersects $\mathbb{D}(t)$ transversely. Thus, for each $t \in [t_\asymp, \overline{T})$, at least one of those curves is not a singleton. Since $\mathbb{D}(t)$ is a disk with radius $\lambda^{-1}(t) \xrightarrow[t \to \overline{T}]{} 0$, the maximum curvature on $M_n(t) \cap \mathbb{D}(t)$ blows up no later than time $\overline{T}$. $\square$

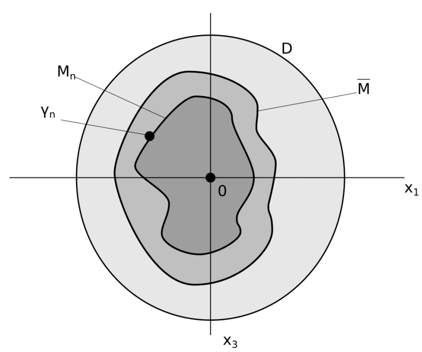

Figure 5.8: Cross section of $K(t)$ at $x_2 = 0$

Now for the case where $\overline{M}(t_\asymp) \setminus \overline{M}_\asymp(t_\asymp)$ is two components.

Proposition 5.25. *Let $\overline{M}_0$ be a smoothly embedded, closed, mean-convex surface. Let M_{n0} be a sequence of smoothly embedded, closed surfaces such that $M_{n0} \to \overline{M}_0$ and $M_{n0} \subset \mathbf{\Omega}_0$. Assume $\overline{M}_0$ is $\overline{M}(t_\asymp) \setminus \overline{M}_\asymp(t_\asymp)$ is* two components, *$\overline{M}$ is type-I with constant C_0, and that $\overline{M}$ has a cylindrical singularity at time $\overline{T}$.*

Then there is an $n_0 > 0$ so that $T_n \leq \overline{T}$ whenever $n > n_0$.

The proof is a rather technical procedure, so we begin with some motivation. Recall the hope is to construct something like that in Figure 5.1. For each sphere, we need to choose the radius r, the time t_0 at which to place the sphere, n so that $M_n(t_0)$ is close to $\overline{M}(t_0)$, and the point $y \in M_n(t_0)$ at which we place the sphere (see Figure 5.10) . When placing the sphere in $M_n(t_0)$, we have three concerns. Note the topology of each bulb has no impact on the method.

(i) *The sphere must fit in $M_n(t_0)$.* Here, we turn to the regularity results of §5.2.4 to apply the Andrews condition. In addressing this, we prescribe a maximum radius for the sphere, thus fixing its lifespan.

(ii) *The sphere should not intersect the neck.* We want that the sphere stays out of K, keeping some points of $M_n(t_0)$ away from the neck. (The neck and sphere will contract away from each other, so this condition is preserved.)

(iii) *The sphere must outlive the neck.* Given the fixed lifespans of $\overline{M}$ and the sphere (once r is chosen), we need only wait to place the sphere until it will live past $\overline{T}$.

The preceding conditions are mostly about $M_n(t_0)$, but we only have control over $M_n(t_0)$ via $\overline{M}(t_0)$ by well-posedness. This makes dependencies more delicate. Therefore the proof is broken into three parts: choosing r, then t_0, then n and y.

In the first part, we choose the radius r small enough to facilitate (ii) and (i). In the second part, we choose t_0 close enough to $\overline{T}$ that (iii) is satisfied and $R(t_0)$ satisfies (ii) (intuitively, we are waiting for $\overline{M}(t)$ to develop a neck very small compared to the bulbs). In the third part, we choose n large enough that $M_n(t_0)$ approximates both bulbs, so a sphere of radius r can be placed in each of $\Omega_{n=\!\circ}(t_0)$ and $\Omega_{n\circ\!=}(t_0)$.

Proof of Proposition 5.25. Assume, without loss of generality, that the cylindrical singularity in question is at the origin. Assume the axis of the cylinder is the x_2-axis, so we can make easy use of $K(t)$.

Assume $t \geq t_{\asymp}$, so $\overline{M}$ has a neck. For simplicity, we'll do the proof just in terms of the right bulb.

By Corollary 5.23, there are $x \in M^*_{\circlearrowleft}$, $p \in \mathcal{M}$, and $C > 0$ so that $\overline{\mathbf{F}}(p,t) \xrightarrow[t \to \overline{T}]{} x \neq 0$ and $\overline{H}(\overline{\mathbf{F}}(p,t)) < C$ for $t \in [t_{\asymp}, \overline{T})$. Several choices of constants and objects in the proof rely on careful spacing with respect to x and the origin. To that end, the quantity $\delta = \frac{|x|}{8}$ is convenient. (See Figure 5.9 for a preview)

Part I: Choosing r Since $\overline{M}_0(t)$ becomes strictly mean convex immediately, there is some $\alpha > 0$ so that $\overline{M}(t_{\asymp})$ is 2α-non-collapsed (so $M_n(t_0)$ will be α-non-collapsed when we choose it). Take

$$r_1 = \frac{\alpha}{2C}$$

as an upper bound for r. Now choose $r = \min\{r_1, \delta\}$.

Later, $r \leq \delta$ will help us with (ii) (see again Figure 5.9), and $r \leq r_1$ will allow us to use well-posedness to help with (i) (we cannot choose *where* to place the sphere until after we have chosen t_0, n, and y).

Part II: Choosing t_0 Since $|\overline{\mathbf{F}}(p,t)| \xrightarrow[t \to \infty]{} x$, there must be $t_1 \in [t_{\asymp}, \overline{T})$ after which $|\overline{\mathbf{F}}(p,t)| \geq \frac{|x|}{2}$, by continuity of the flow. Let $R(t) = 2C_0\lambda^{-1}(t) = \min_{z \in K(t)} |z|$. Then $R(t)$ shrinks to 0 by time $\overline{T}$. Thus there is a time $t_2 \in [t_1, \overline{T})$ after which $R(t) \leq \frac{|x|}{16}$ (See Figure 5.9). These two conditions will help with (ii) in part III.

Since the radius r of the sphere is already fixed, we know its lifespan. Call it τ. We can find the time $t_3 \in [t_2, \overline{T})$ at which $\tau = 2(\overline{T} - t_3)$. This means if the sphere begins its flow at any time in $[t_3, \overline{T})$, the sphere will survive past time $\overline{T}$.

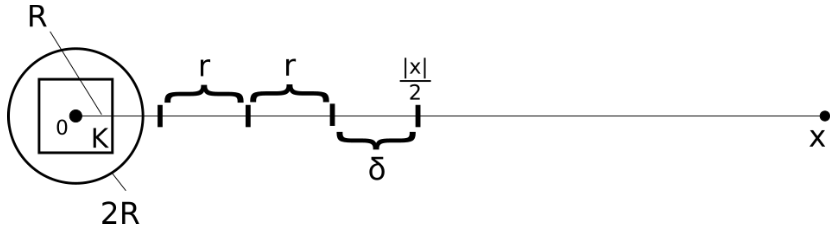

Figure 5.9: "Worst case scenario", with distances aligned on the x_2-axis

Choose $t_0 = t_3$, so t_0 has the properties of t_1, t_2, t_3. Then, because of the choice of t_3, we have already satisfied (iii).

Part III: Choosing n and y Let $x_0 = \overline{\mathbf{F}}(p, t_0)$, so $|x_0| \geq \frac{|x|}{2}$.

By well-posedness, there is n_1 such that if $n \geq n_1$, $M_n(t_0)$ is α-non-collapsed. Given $\delta > 0$ above, and recalling $\overline{H}(x_0) = \overline{H}(\overline{\mathbf{F}}(p, t_0)) < C$, there exists $n_2 \geq n_1$ so large that, if $n \geq n_2$, then there is a point $y \in M_n(t_0)$, within δ of x_0, so that $H_n(y) \leq 2C$. Let $n_0 = n_2$. Now assume $n \geq n_0$ so that $M_n(t_0)$ is α-non-collapsed and that such a y exists. Choose that y.

We now have the following:

$$|y| \geq |x_0| - \delta \geq \tfrac{|x|}{2} - \delta = 3\delta$$
$$R(t_0) \leq \tfrac{\delta}{2}$$
$$\text{sphere diameter} = 2r \leq 2\delta.$$

Those together imply that, were a sphere of radius r placed at time t_0 touching y, the distance between sphere and $K(t_0)$ is at least $\frac{\delta}{2}$. Since the sphere would contract under mean curvature flow, and $K(t)$ contracts by definition, they would stay disjoint. Thus if we flow the sphere by mean curvature flow, as we do with $M_n(t)$, we are done with (ii). (See Figures 5.10 and 5.9)

Recall $M_n(t)$ is α-non-collapsed. Then, since $H_n(y) \leq 2C < \infty$, and $r \leq r_1 = \frac{\alpha}{2C}$, there is room to place a sphere of radius r inside $Cl(\Omega_n(t_0))$, tangent at y. Since the sphere is

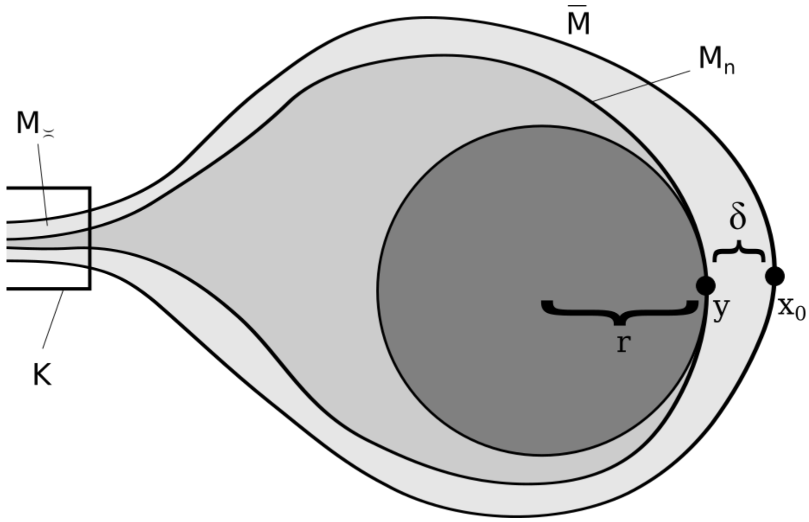

Figure 5.10: Sphere fits in bulb far from neck

disjoint from $K(t_0)$, it is contained in $Cl(\Omega_{n\supset}(t_0))$. That takes care of (i).

Thus, for our choice of r, t_0, n, and y, (i)-(iii) are all satisfied, with regards to the right bulb.

Finishing the proof Repeat the above argument for $\overline{M}_{\subset}(t)$.

Since there is a sphere in each bulb of $\Omega_n(t)$, up to time $\overline{T}$, there are points of $M_n(t)$ in each bulb of $\mathbf{\Omega}_n(t)$ up to time $\overline{T}$. Therefore, if $T_n > \overline{T}$, then $M_n(t)$ has a nontangential intersection with $\mathbb{D}(t)$ for $t \in [t_0, \overline{T})$. As in the proof of Proposition 5.24, since $\overline{M}(t) \cap \mathbb{D}(t)$ is a closed curve, and $M_n(t) \subset \mathbf{\Omega}(t)$, at least one component of $M_n(t) \cap \mathbb{D}(t)$ is a nontrivial closed curve for all $t \in [t_0, \overline{T})$. Since $\mathbb{D}(t)$ collapses to a point at time $\overline{T}$, the maximum curvature on $M_n(t) \cap \mathbb{D}(t)$ must blow up.

□

Propositions 5.24 and 5.25 completely cover the case where $M_{n0} \subset \mathbf{\Omega}_0$. So we mimic the proof of Theorem 5.1 to reduce the proof of Theorem 5.2 to that case.

Proof of Theorem 5.2. Let $\overline{M}$ and M_n be as in Theorem 5.2. By well-posedness, we already have that $\liminf_{n\to\infty} T_n \geq \overline{T}$. So we need only show that $\limsup_{n\to\infty} T_n \leq \overline{T}$.

Let $0 < \varepsilon < \frac{T_n}{2}$. Define $\widehat{M}_n(t) = M_n(t+\varepsilon)$ so the smooth existence time interval for $\widehat{M}_n$ is $[-\varepsilon, T_n - \varepsilon)$. (We want $\varepsilon < \frac{T_n}{2}$ so the smooth existence time intervals for $\overline{M}$ and $\widehat{M}_n$ overlap by more than $\frac{\varepsilon}{2}$.)

Now $\widehat{M}_{n0} = M_n(\varepsilon)$. By the minimum principle for mean curvature, $\overline{M}(t)$ strictly is mean-convex for $t = [\frac{\varepsilon}{2}, \overline{T})$. Therefore its velocity at every point after time $t = \frac{\varepsilon}{2}$ is inward with positive speed. Thus $\overline{M}_0(\varepsilon) \subset \mathbf{\Omega}_0$, and we have the Hausdorff distance $d := d_H\left(\overline{M}(\varepsilon), \overline{M}_0\right) > 0$. By well-posedness, there is an $n_0 > 0$ so that $d_H(\overline{M}(\varepsilon), \widehat{M}_{n0}) = d_H(\overline{M}(\varepsilon), M_n(\varepsilon)) < \frac{d}{2}$ whenever $n \geq n_0$. (see Lemma 3.1). So assume $n \geq n_0$. Rearranging

$$d_H(\overline{M}(\varepsilon), \overline{M}_0) \leq d_H(\overline{M}(\varepsilon), \widehat{M}_{n0}) + d_H(\widehat{M}_{n0}, \overline{M}_0)$$

gets us

$$d_H(\widehat{M}_{n0}, \overline{M}_0) \geq d_H(\overline{M}(\varepsilon), \overline{M}_0) - d_H(\overline{M}(\varepsilon), \widehat{M}_{n0}) > d - \frac{d}{2} = \frac{d}{2} > 0.$$

Thus $\widehat{M}_{n0} \subset \mathbf{\Omega}_0$.

Now, since multiple results require connectedness, simply choose a component of $M(t)$ that develops a singularity at time $\overline{T}$ and apply all work to it. We turn to Corollary 4.21 to see that $\overline{M}$ must shrink to a point at time $\overline{T}$ or have a cylindrical point at time $\overline{T}$. In the former case, apply Theorem 5.1. In the latter case, we need to apply Proposition 5.24 or Proposition 5.25 accordingly.

Then we see that $T_n = \widehat{T}_n + \varepsilon \leq \overline{T} + \varepsilon$. Since that is true for any $n \geq n_0$, we have $\limsup_{n\to\infty} T_n \leq \overline{T} + \varepsilon$. Since ε was arbitrary, we are done. Then either Proposition 5.24 or Proposition 5.25 applies.

$\square$

5.4 Continuity of the Limit Set

Recall the Hausdorff distance

$$d_H(X,Y) = \max\left\{\sup_{x\in X}\inf_{y\in Y}|x-y|, \sup_{y\in Y}\inf_{x\in X}|y-x|\right\},$$

and that if $\widehat{\Sigma}$ is a graph of f over Σ, then by Lemma 3.1

$$d_H(\Sigma,\widehat{\Sigma}) \le \|f\|_{C^{)}}.$$

Remark 5.26. *Although the case where $\overline{M}$ contracts to a point can be made very simple with an argument similar to that in the proof of Theorem 5.1, the proof of Corollary 5.3 below suffices for both spherical and cylindrical cases.*

Proof of Corollary 5.3 For intuition, note from parabolic regularity (under the assumption that $T_n > \overline{T}$), it makes sense that

$$M_n(\overline{T}) \sim M_n(t) \sim \overline{M}(t) \sim \overline{M}(\overline{T})$$

for a fixed t close to, but less than, $\overline{T}$. We need Theorems 5.1 and 5.2 to assert that $M_n^* = M_n(T_n)$ is anything like $M_n(\overline{T})$.

Proof. Let $\varepsilon > 0$, and set $C = \sqrt{2N}$. Let d_H denote Hausdorff distance. Lemma 3.1 says we can make $d_H\left(M_n(t_0), \overline{M}(t_0)\right)$ small by making n large.

Choose:

- $t_0 \in [0,\overline{T})$ so that $|\overline{T} - t_0| < \varepsilon$.

- n_1 so $n \ge n_1$ implies $|\overline{T} - T_n| < \frac{\varepsilon}{2}$ (which exists by Theorem 5.1 or 5.2).
  (Keeps T_n close to $\overline{T}$, which also means $T_n > t_0$.)

- n_2 so $n \geq n_2$ implies $d_H(M_n(t_0), \overline{M}(t_0)) < \varepsilon$ (which exists by well-posedness).

By Lemma 3.3, every point of $\overline{M}(t_0)$ is within $C\sqrt{\overline{T} - t_0}$ of $\overline{M}^*$. Therefore, we have

$$d_H\left(\overline{M}(t_0), \bar{M}^*\right) \leq C\sqrt{\overline{T} - t_0} \leq C\sqrt{\varepsilon}$$

and

$$\begin{aligned} d_H(M_n^*, M_n(t_0)) &\leq C\sqrt{T_n - t_0} = C\sqrt{(T_n - \overline{T}) + (\overline{T} - t_0)} \\ &\leq C\left(\sqrt{|T_n - \overline{T}|} + \sqrt{|\overline{T} - t_0|}\right) < 2C\sqrt{\varepsilon}. \end{aligned}$$

Now assume $n \geq \max\{n_1, n_2\}$, and apply

$$\begin{aligned} &d_H(M_n^*, \bar{M}^*) \\ \leq &d_H(M_n^*, M_n(t_0)) + d_H(M_n(t_0), \overline{M}(t_0)) + d_H(\overline{M}(t_0), \bar{M}^*) \\ \leq &2C\sqrt{\varepsilon} + \varepsilon + C\sqrt{\varepsilon}. \end{aligned}$$

Since ε was arbitrary, we are done. $\square$

Chapter 6

Liouville Theorem

6.1 Introduction

We study ancient solutions to mean curvature flow. Let $\mathbf{F} : \mathcal{M} \times \mathbb{R}^- \to \mathbb{R}^{N+1}$ be a family of smooth embeddings $\mathbf{F}(\cdot, t) = M(t)$, where $\mathcal{M}$ is a closed N-dimensional manifold. We say that $M = \{M(t)\}_{t \in [0,T)}$ is a mean curvature flow if

$$\partial_t \mathbf{F} = -H\nu, \tag{6.1}$$

where H is the scalar mean curvature, ν is the *outward* unit normal, and $-H\nu$ is the mean curvature vector.

We call a mean curvature flow *ancient* if it is defined for all negative time. Ancient solutions arise as blow-ups of singularities. Daskalopoulos, Hamilton, and Šešum completely classified ancient convex solutions for embedded curves in $\mathbb{R}^2$ in [13]. Here our goal is to further the classification to two dimensions for mean-convex, type-I, non-collapsed flows. At any point in time, an ancient solution has had an arbitrarily long amount of time for diffusion to take place, so we expect it to be highly regular and symmetric. We see this in the work of Huisken and Sinestrari in [30] where they show, assuming convexity and compactness, a number of conditions equivalent to the flow being a shrinking sphere. This is similar to our result here, so we emphasize that although we impose other restrictions, we allow for compactness or noncompactness. (Haslhofer and Kleiner show in [25] that ancient mean-convex, non-collapsing solutions are convex anyway.)

In the theorem, we do assume some regularity to begin with. In the spotlight are the type-I curvature bound and the non-collapsing condition. With the type-I assumption, we show that for an eternal solution for the rescaled flow, all orders of curvature are bounded in time. The non-collapsing condition prevents sheeting, thereby preserving embeddings as $t \to -\infty$. This is important for integral convergence if one intends to integrate on the embedded hypersurface itself, rather than a background manifold. Both assumptions are rather strong, but since we have in mind ancient solutions which arise from blow-ups at singularities of type-I, mean-convex, compact flows, both are quite reasonable.

There are examples of ancient solutions that do not satisfy the conclusions of our main theorem. The paperclip solution, one of the two classes in [13], converges to two parallel lines as $t \to -\infty$, but behaves like the grim reaper solution at either end. This was generalized in a sense by White in [43] to higher dimensions, but was studied in more detail by Haslhofer and Hershkovitz in [24]. The paperclip, however, is neither type-I, nor non-collapsing, as $t \to -\infty$.

The method here is inspired by that of Giga and Kohn in [22]. There they show that the rescaled limits as $t \to -\infty$ and $t \to +\infty$ are the same. They then classify self-similar solutions to find that the forward and backward limits of the rescaled solution must have the same energy. The energy they use is decreasing, so once they relate it to the time derivative of the solution, they can integrate across time to show the the solution is constant in time.

We can build off the work of Huisken in [28] or White in [43] to classify the forward limit, and the work of Haslhofer and Kleiner in [25] to classify the backward limit. However, the geometric nature of the flow adds a complication: there are different self-similar solutions that can arise as blow-ups and blow-downs, and they have different energies. We calculate the energy (Huisken's Gaussian area functional defined in [28]) explicitly in each case. The fact that energy is decreasing means that the backward limit cannot have a lower energy than the forward limit, but this does not cover the case when the backward limit has a strictly higher energy than that of the forward limit. We see in the proof of Proposition 6.14 that the only case in which the monotonicity does not help is a noncompact backward limit with a compact forward limit. This case is ruled out rather directly in Lemma 3.5, since the rescaled evolution equation tends to expand the hypersurface.

We now give some definitions so we can state the main theorem (Theorem 6.3).

Definition 6.1 (Type-I Flow)**.** *Let M be a mean curvature flow for times $t \in \mathbb{R}^-$. Let $\lambda(t) = (-2t)^{-\frac{1}{2}}$, and write A for the second fundamental form. We say M is type-I if there is a $C_0 > 0$ so that*

$$\max_{x \in M(t)} |A(x,t)| \leq C_0 \lambda(t) \text{ for } t \in \mathbb{R}^-.$$

Remark 6.2. *The type-I condition is typically employed in discussions of blow-ups at singularities. However we apply the condition to the entirety of an ancient flow, meaning curvature decays as $t \searrow -\infty$ as well.*

Theorem 6.3 (Main Theorem)**.** *Let $M(t)$ be a smooth, properly embedded, complete, ancient, type-I, mean-convex, α-non-collapsed, two-dimensional mean curvature flow in $\mathbb{R}^3$ with first singular point x at time $t = 0$. Further assume that $M(t)$ has uniform polynomial volume growth on $\mathbb{R}^-$.*

Then $M(t)$ is either a sphere or cylinder, shrinking homothetically until it vanishes at time $t = 0$.

Remark 6.4. *The assumption that $N = 2$ is necessary to restrict the topologies of blow-ups at a singular point. See Proposition 6.14 and the discussion before it for further explanation.*

Remark 6.5. *Of course, for manifolds without boundary, properly embedded implies completeness. Furthermore, we have by Corollary 4.7 that α-non-collapsed implies properly embedded and uniform polynomial volume growth.*

6.2 Some Technical Lemmas

Lemma 6.6 (Proposition 2.3 of [28]). *Given $s_0 \in \mathbb{R}$ (and corresponding t_0), for each $m > 0$, there is $C(m) < \infty$, such that $|\widetilde{\nabla}^m \widetilde{A}|^2 < C(m)$ holds on $\widetilde{M}(s)$ uniformly in s, where $C(m)$ depends on N, m, C_0, and $M(t_0)$.*

This phrasing is changed slightly to accomodate ancient solutions by choosing $M(t_0)$ as "initial data".

Lemma 6.7 (Corollary 3.2 of [28]). *For the rescaled flow $\widetilde{M}$,*

$$\partial_s \widetilde{E}(s) = \int_{\widetilde{M}(s)} \left| \widetilde{\mathbf{F}}^{\perp} - \widetilde{H}\widetilde{\nu} \right|^2 \widetilde{\rho}\, d\widetilde{\mu}.$$

Lemma 6.8. *For a mean curvature flow M and rescaled flow $\widetilde{M}$, $\widetilde{E}(s) = (2\pi)^{\frac{N}{2}} E(t)$.*

The proof is a direct calculation.

6.3 Regularity

We will need two time derivatives of E later, which involves fourth order terms, so we need high-regularity control to properly manage convergence. Huisken takes care of this forward in time in [28], but the proof relies on a maximum principle. We need to prove bounds for $|\widetilde{\nabla}^m \widetilde{A}|$ backward as well. We refer to a parabolic regularity result in [14]. (See Figure 6.1)

Lemma 6.9 (Proposition 3.22 of [14]). *Let (M_t) be a smooth, properly embedded solution of mean curvature flow in $B_\rho(x_0) \times (t_0 - \rho^2, t_0)$*

$$|A(x)|^2 \leq \frac{c_0}{\rho^2}$$

for all $t \in (t_0 - \rho^2, t_0)$ and $x \in M_t \cap B_\rho(x_0)$. Then for every $m \in \mathbb{N}$ there is a constant $c_m = c_m(N, m, c_0)$ such that for all $x \in M_t \cap B_{\frac{\rho}{2}(x_0)}$ and $t \in (t_0 - \frac{\rho^2}{4})$,

$$|\nabla^m A(x)|^2 \leq \frac{c_m}{\rho^{2(m+1)}}.$$

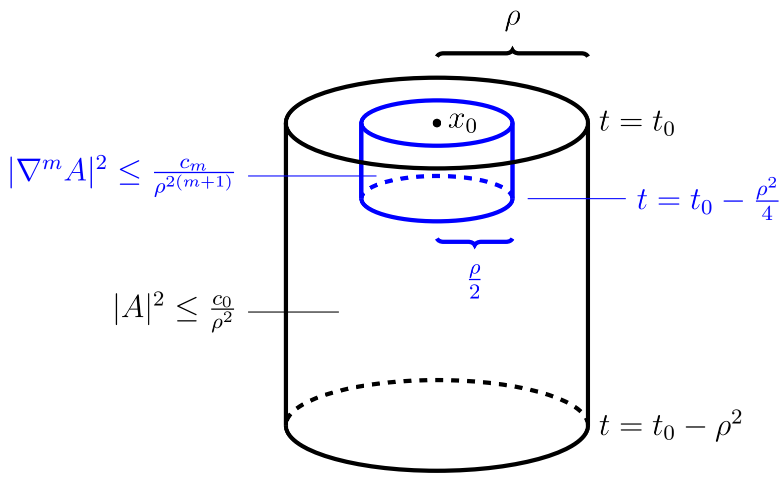

Figure 6.1: Small parabolic ball inside larger parabolic ball.

Now we want to use the above lemma to get a bound on covariant derivatives in the rescaled flow, and we want to do so for all time. For $s > 0$, Huisken did this in Proposition 2.3 of [28] (Lemma 6.6 of this work). For $s < 0$, we take advantage of the type-I bound. In the nonrescaled setting, going farther back in time forces the curvature to decay. This allows us to choose larger ρ for more control on $|\nabla A|$.

Lemma 6.10 (Ancient Regularity)**.** *Let M be a smooth, properly embedded, ancient, type-I mean curvature flow. Then for $m \in \mathbb{N}$, there is $c_m > 0$ so that for each $t \in \mathbb{R}^-$,*

$$|\nabla^m A| \le \sqrt{c_m}\lambda^{m+1}(t) = \sqrt{c_m}(-2t)^{-\frac{m+1}{2}}$$

uniformly over $M(t)$.

Proof. Let $t \in \mathbb{R}^-$, and $x \in M(t)$. With the goal of applying Lemma 6.9, choose $\rho = \lambda^{-1}(t) = (-2t)^{-\frac{1}{2}}$, $x_0 = x$, and $t_0 = t + \frac{\rho^2}{8} = \frac{3}{4}t < 0$. That puts our point of interest, $(x_0, t) = (x, t)$, at the center of the inner cylinder, with t_0 at the top of the cylinders.

Due to the type-I bound, $|A(y,\tau)| \leq C_0\lambda(t_0)$ for every (y,τ) in the outer cylinder since $\tau \leq t_0$. Now setting $c_0 = \frac{2}{\sqrt{3}}C_0$,

$$|A(y,\tau)| \leq C_0\lambda(t_0) = C_0\lambda\left(\frac{3}{4}t\right) = C_0\frac{2}{\sqrt{3}}\lambda(t) = \frac{c_0}{\rho}.$$

Now recall (x,t) is in the inner cylinder. Then since $|A| \leq \frac{c_0}{\rho^2}$ in the outer cylinder, Lemma 6.9 says that for every $m \in \mathbb{N}$, there is c_m so $|\nabla^m A|^2 \leq \frac{c_m}{\rho^{2(m+1)}}$ in the inner cylinder. Rather,

$$|\nabla^m A(x,t)| \leq \frac{\sqrt{c_m}}{\rho^{m+1}} = \sqrt{c_m}\lambda^{m+1}(t).$$

$\square$

Corollary 6.11 (Eternal Regularity)**.** *Let M be a smooth, properly embedded, ancient, type-I mean curvature flow. Then for $m \in \mathbb{N}$, there is $C_m > 0$ so that*

$$\sup_{\xi\in\widetilde{M}(s), s\in\mathbb{R}} |\widetilde{\nabla}^m\widetilde{A}| \leq C_m.$$

Proof. Recall $\lambda(t) = \frac{e^s}{\sqrt{2}}$, so that we have from Lemma 6.10,

$$|\widetilde{\nabla}^m\widetilde{A}| = \lambda^{1-m}|\nabla^m A| \leq \sqrt{c_m}\lambda^{1-m}\lambda^{m+1} = \sqrt{c_m}\lambda^2 = \frac{\sqrt{c_m}}{2}e^{2s}.$$

Now, for $s \in (-\infty, 0)$, $|\widetilde{\nabla}^m\widetilde{A}| \leq \frac{\sqrt{c_m}}{2}$. Then Lemma 6.6 provides a $C_m \geq \frac{\sqrt{c_m}}{2}$ for which $|\widetilde{\nabla}^m\widetilde{A}| \leq C_m$ for $s \in (0,\infty)$. Therefore,

$$|\widetilde{\nabla}^m\widetilde{A}| \leq C_m$$

for all time. $\square$

6.4 Proving the Main Theorem

Theorem 6.12 (Subsequential Limits)**.** *Let M be a smooth, properly embedded, ancient, type-I, mean-convex, α-non-collapsed, two-dimensional mean curvature flow with uniform polynomial growth. Assume M has a singular point at the origin at time $t = 0$.*

Then for every sequence of rescaled times $s_i \searrow \infty$, there is a subsequence $\{s_{i_j}\}$ so that $\lim_{j\to\infty} \widetilde{M}(s_{i_j})$ converges to some $\widetilde{M}_{-\infty}$ in C^2_{loc} in the graph sense. Furthermore, $\widetilde{M}_{-\infty}$ is either a plane passing through 0, a cylinder centered at 0 with radius 1, or a sphere centered at 0 with radius $\sqrt{2}$.

All the same can be said of some sequence $s_i \nearrow \infty$ and a limit $\widetilde{M}_{+\infty}$. Although in that case we can rule out the plane.

Proof. Since $|\nabla^m A| \leq C_m$ by Corollary 6.11 and $\widetilde{M}(s) \cap B_N(0)$ is nonempty by Corollary 3.6, $\widetilde{M}_{-\infty}$ exists by Corollary 4.7. We know from (2.4) that $\widetilde{M}_{-\infty}$ is a tangent flow, or blowdown soliton. Therefore Theorem 1.11 of [25] says that $\widetilde{M}_{-\infty}$ is either a plane, cylinder, or sphere.

Again, $\widetilde{M}_{+\infty}$ exists due to Corollary 4.7. Since $\widetilde{M}_{+\infty}$ is a tangent flow, we know it is either a plane, cylinder, or sphere by Theorem 1 of [43]. However, Corollary 1.8 of [40] rules out the plane for tangent flows at first singularities for mean-convex flows.

It follows from the rescaled flow equation that for a stationary sphere or cylinder, $\widetilde{H} = \widetilde{\mathbf{F}} \cdot \widetilde{\nu}$. The necessary radii follow directly from there.

□

Lemma 6.13. *Let M be as in Theorem 6.12. The limits $\widetilde{E}_{\pm\infty} := \lim_{s\to\pm\infty} \widetilde{E}(s)$ exist. Furthermore, the limits $\widetilde{E}_{\pm\infty}$ are equal to the Gaussian areas of $\widetilde{M}_{\pm\infty}$.*

Proof.

The Limits $\widetilde{E}_{\pm\infty}$ Exist Since $M(t)$ exhibits uniform polynomial volume growth, $E(t)$ is bounded for $t \in \mathbb{R}^-$. Then by Lemma 6.8, $\widetilde{E}(s)$ is also bounded for all $s \in \mathbb{R}$. We know from Lemma 6.7 that $\widetilde{E}$ is decreasing in time and bounded below by 0. Therefore, its limits at times $\pm\infty$ both exist. We denote them $\widetilde{E}_{\pm\infty}$.

Gaussian areas We do the proof for $\widetilde{M}_{-\infty}$, and the proof for $\widetilde{M}_{+\infty}$ is identical. One will notice below that different radii $R+\varepsilon$ and R are used in the domains for integrals. This is of little interest, but necessary to accomodate the normal vectors to $\widetilde{M}_{-\infty} \cap B_R(0)$, which leave the ball near the boundary.

Let $0 < \varepsilon < 1$. By uniform polynomial volume growth, there exists $R > 0$ such that

$$\int_{\widetilde{M}(s_i)\setminus B_R(0)} \widetilde{\rho}\, d\widetilde{\mu}_i < \varepsilon$$

for all i and also for $\widetilde{M}(s_i)$ replaced by $\widetilde{M}_{-\infty}$. By Corollary 4.7, for large i there are open $V_i \subset \widetilde{M}_{-\infty} \cap B_{R+\varepsilon}(0)$ and $f_i : V_i \to \mathbb{R}$ with $\|f_i\|_{C^1} < \varepsilon$ such that

$$\varphi_i(x) := x + f_i(x)\widetilde{\nu}_{-\infty}(x)$$

is a diffeomorphism from V_i onto $\widetilde{M}(s_i) \setminus B_R(0)$.

Then

$$\int_{\widetilde{M}(s_i)\cap B_R(0)} \widetilde{\rho}\, d\widetilde{\mu}_i = \int_{\widetilde{M}_{-\infty}\cap B_{R+\varepsilon}(0)} \chi_{V_i}\widetilde{\rho}(\varphi_i(x))\sqrt{1+|\widetilde{\nabla}_{-\infty} f_i|^2}\, d\widetilde{\mu}_{-\infty},$$

where the integrals now have a fixed domain, and χ_{V_i} is the characteristic function The integrand is bounded by 2 and converges pointwise to $\widetilde{\rho}$. Therefore we can apply dominated convergence. Taking i large enough, and repeatedly absorbing $O(\varepsilon)$-terms, we write

$$\begin{aligned}
\int_{\widetilde{M}(s_i)} \widetilde{\rho}\, d\widetilde{\mu}_i &= \int_{\widetilde{M}(s_i)\cap B_R(0)} \widetilde{\rho}\, d\widetilde{\mu}_i + O(\varepsilon) \\
&= \int_{\widetilde{M}_{-\infty}\cap B_{R+\varepsilon}(0)} \widetilde{\rho}\, d\widetilde{\mu}_{-\infty} + O(\varepsilon) \\
&= \int_{\widetilde{M}_{-\infty}} \widetilde{\rho}\, d\widetilde{\mu}_{-\infty} + O(\varepsilon),
\end{aligned}$$

where we used

$$\int_{\widetilde{M}_{-\infty}\setminus B_{R+\varepsilon}(0)} \widetilde{\rho}\, d\widetilde{\mu}_{-\infty} \leq \int_{\widetilde{M}_{-\infty}\setminus B_R(0)} \widetilde{\rho}\, d\widetilde{\mu}_{-\infty}.$$

□

The following result is where we really need $N = 2$. That is, if $\widetilde{M}_{-\infty}$ and $\widetilde{M}_{+\infty}$ can be generalized cylinders, our method does not prevent them from being generalized cylinders with different numbers of flat factors. Lemma 3.5 lets us handle the case where either limit is a (compact) sphere, and we are able to rule out planes altogether. Restricting our scope to surfaces means the only other possibility is cylinders with the known factorization $\mathbb{S}^1 \times \mathbb{R}^1$.

Proposition 6.14 ($\widetilde{M}_{-\infty} \cong \widetilde{M}_{\infty}$)**.** *Let M be a smooth, complete, properly embedded, ancient, type-I, mean-convex, α-non-collapsed, two-dimensional mean curvature flow with uniform polynomial volume growth. Assume M has a singular point at the origin at time $t = 0$.*

Then $\widetilde{M}_{-\infty}$ and $\widetilde{M}_{\infty}$ are either both spheres or are both cylinders. They have the same radius, and are centered at the origin.

Proof. Recall from Theorem 6.12 we know that $\widetilde{M}_{-\infty}$ is either a plane, cylinder, or sphere, and $\widetilde{M}_{+\infty}$ is only a cylinder or sphere.

Now we turn our attention to determining possible shapes for $\widetilde{M}_{-\infty}$. The strategy is to use the monotonicity of $\widetilde{E}$ to rule out the plane, then show that $\widetilde{M}_{-\infty}$ if and only if $\widetilde{M}_{\infty}$. If $\widetilde{E}_P$, $\widetilde{E}_C$, and $\widetilde{E}_S$ are the Gaussian areas for the plane, cylinder of radius 1, and sphere of radius $\sqrt{2}$ respectively, a direct calculation gives $\widetilde{E}_P = 2\pi$, $\widetilde{E}_C = 2\pi\sqrt{\frac{2\pi}{e}}$, and $\widetilde{E}_S = 2\pi\frac{4}{e}$. That is

$$\widetilde{E}_P < \widetilde{E}_S < \widetilde{E}_C.$$

First suppose $\widetilde{M}_{-\infty}$ is a plane. We already know $\widetilde{M}_{\infty}$ is a cylinder of radius 1 or a sphere of radius $\sqrt{2}$. However that would mean $\widetilde{E}$ increased, which is a contradiction.

If either $\widetilde{M}_{-\infty}$ or $\widetilde{M}_{\infty}$ is a sphere, then there is $s \in \mathbb{R}$ so that $\widetilde{M}(s)$ is compact. Thus by Lemma 3.5, $\widetilde{M}$ is a compact flow. Therefore both $\widetilde{M}_{-\infty}$ and $\widetilde{M}_{\infty}$ must be the same sphere.

Now we have that $\widetilde{M}_{+\infty}$ is a sphere if and only if $\widetilde{M}_{-\infty}$ is a sphere. Then, by process of elimination, $\widetilde{M}_{+\infty}$ is a cylinder if and only if $\widetilde{M}_{-\infty}$ is a cylinder. Thus $\widetilde{M}_{+\infty}$ must be isometric to $\widetilde{M}_{-\infty}$, since Theorem 6.12 ensures they have the same radius. Due to the equations $\widetilde{\mathbf{F}}_{\pm\infty} \cdot \widetilde{\nu}_{\pm\infty} = \widetilde{H}_{\pm\infty}$, the sphere or cylinder must be centered around the origin. $\square$

Finally, since $\widetilde{M}_{+\infty}$ and $\widetilde{M}_{-\infty}$ are isometric and both centered at 0, they have the same Gaussian area. However, the axis of $\widetilde{M}_{-\infty}$ could depend on the subsequence. We address this issue in the following proposition.

Proposition 6.15. *Let M be as in Proposition 6.14. Then $\widetilde{M}_{-\infty} \equiv \widetilde{M}(s) \equiv \widetilde{M}_{+\infty}$.*

Proof. From Proposition 6.14, $\widetilde{M}_{-\infty}$ and $\widetilde{M}_{+\infty}$ are isometric. Then we can write

$$0 = \widetilde{E}_{-\infty} - \widetilde{E}_{\infty} = \int_{\infty}^{-\infty} \int_{\widetilde{M}(s)} \left| \widetilde{\mathbf{F}}^{\perp} - \widetilde{H}\widetilde{\nu} \right|^2 \widetilde{\rho}\, d\widetilde{\mu}\, ds$$

Thus we conclude that $\left(\partial_s \widetilde{\mathbf{F}}\right)^{\perp} = \widetilde{\mathbf{F}}^{\perp} - \widetilde{H}\widetilde{\nu} = 0$ for all time. This means, up to tangential diffeomorphism, that $\widetilde{M}(s)$ is stationary. Thus $\widetilde{M}(s)$ is a fixed sphere or cylinder.

$\square$

Proof of Main Theorem. Without loss of generality, assume M has a singularity at (0,0). By Proposition 6.14 and Proposition 6.15, $\widetilde{M}(s)$ is either a stationary sphere or cylinder centered at the origin. This corresponds to a homothetically shrinking $M(t)$ that is a sphere or cylinder. $\square$

Chapter 7

Conclusion

The eventual goal of this program is to show some universality to the profile near a type-I neck-pinch similar to [21] or [1], as in [33]. There are three major steps to this:

1. Show the set of initial conditions leading to type-I singularities is open
2. Use our expression of a neck to decompose the problem into a finite-dimensional dynamical system, since there is a small number of nondecaying eigenmodes.
3. Show uniform asymptotics for those eigenmodes.

This dissertation constitutes two thirds of that first step and I have made substantial progress on the rest, though it is yet unclear to me whether finishing the first step will require completely new theory. All of Chapter 5 is analogous to only three pages of [33], so the problem may be quite complex.

Furthermore, Chapter 5 and Chapter 6 both only apply to two dimensions for different reasons. I believe the blow-up time continuity argument can be generalized to higher dimensions if two-convexity is assumed. That is, the sum of the lowest eigenvalues of the second fundamental form is nonnegative so that a neck still forms, as in [30]. Unfortunately it will require a deeper understanding of the topology of M^* near the singularity, and is probably the end of the road for the neck-pinching technique.

It should be noted that beyond the use of previous results, most of the work herein is incredibly elementary. To my knowledge, standard abstract approches have failed to make progress on the blow-up continuity for mean curvature flow. Perhaps a heavy reliance on the basics can continue to open doors to new results.

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Appendices

A Barebones Geometric Primer

Here we have a very minimal account of the basic geometric ideas needed to define mean curvature flow. For simplicity, this is done entirely in the context of Euclidean space. The purpose of this appendix is just to show that the mean curvature and its evolution are indeed definite, sensible, and calculable, since the matter is not addressed or needed for the main work beyond the scaling property of the mean curvature H. The calculations herein are from Appendices A and B of [14]. For a thorough investigation of evolution equations of the following quantities for mean curvature flow, see [37].

Let $\mathcal{M}$ be an abstract, closed N-dimensional manifold. Let $\mathbf{F} : \mathcal{M} \to \mathbb{R}^{N+1}$ be a smooth embedding and $M = \mathbf{F}(\mathcal{M})$ (assume the orientation of $\mathbf{F}$ is such that the Jacobian J is positive so that the outward *unit normal* ν is consistent with choice of orientation).

With a coordinate system $\{p_i\}_{i=1}^N$ on $\mathcal{M}$, denote the i-th tangent vector to M at a point $x = \mathbf{F}(p)$ by $\partial_i \mathbf{F}(p) := \frac{\partial \mathbf{F}}{\partial p_i}(p)$. This way we can later express $\mathbf{F}$ as

$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}^i \partial_i \mathbf{F} + \mathbf{F}^{N+1} \nu.$$

We also write $\partial_i f := (\partial_i \mathbf{F}) f$ for the directional derivative of a function f on M.

Henceforth we use Einstein summation notation. The components of vectors have upper indices and the coordinate vectors, acting as derivatives or not, have lower indices. Whenever a term has an upper and lower index that match, it is assumed that index is summed over. For example, we may write for a tangent vector V

$$V = v^i \partial_i F := \sum_i v^i \partial_i F.$$

The tangent space $T_x M$ is the span of the tangent vectors $\{\partial_i \mathbf{F}(p)\}$ (one can think of recentering x to the origin). Then the metric on M induced by $\mathbf{F}$ is the one it inherits from

$\mathbb{R}^{N+1}$. That is, the metric g is a function on M and its tangent spaces giving the dot product of the tangent vectors:

$$g_{ij} = \partial_i \mathbf{F} \cdot \partial_j \mathbf{F},$$

so g can be described by a matrix G of functions g_{ij}. In fact, the Jacobian J of $\mathbf{F}$ can also be written as $J = \sqrt{\det G}$.

It happens that G^{-1} projects onto the tangent space of M, so it can be used to calculate the covariant derivative $\nabla^M f$ of a function f on M, which is like the tangent component of the gradient of f. (We know G^{-1} always exists because we are assuming F is an embedding, so M is N-dimensional. Therefore J is nonzero, and thus so is $\det G$.) That is

$$\nabla^M f = g^{ij} \partial_i f \partial_j \mathbf{F},$$

where g^{ij} are the components of G^{-1} (remember this is summed over i *and* j). That means $(\nabla^M f)^i = \partial_i f$. Since g^{ij} are the components of G^{-1} and $G^{-1}G = I$,

$$g_{ac} g^{cb} = \delta_a^b, \tag{A.1}$$

where δ_j^i is 1 if $i = j$, and 0 otherwise. Considering $V = v^i \partial_i$, we may want to calculate

$$V \cdot \partial_j \mathbf{F} = v^k \partial_k \mathbf{F} \cdot \partial_j \mathbf{F} = g_{jk} v^k.$$

Then we can isolate the coordinate with G^{-1} like

$$g^{ij}(V \cdot \partial_j \mathbf{F}) = g^{ij} g_{jk} v^k = \delta_k^i v^k = v^i.$$

where $v^j = V \cdot \partial_j \mathbf{F}$.

Not only do we write

$$\nabla_{\partial_i \mathbf{F}} f = \partial_i,$$

but the covariant derivative can be used to find vector-valued M-intrinsic directional derivatives of vectors which we describe in terms of projection onto the tangent space:

$$\nabla_{\partial_i \mathbf{F}} \partial_j \mathbf{F} = (\partial_i \partial_j \mathbf{F})^\top =: \Gamma_{ij}^k \partial_k \mathbf{F},$$

where Γ_{ij}^k, the *Christoffel symbols*, are defined by that equation. We want $\nabla_W V$ to be linear in both entries and to follow the product rule in the following sense:

$$\nabla_{\partial_i \mathbf{F}} V = \nabla_{\partial_i \mathbf{F}} (v^k \partial_k \mathbf{F}) = (\nabla_{\partial_i \mathbf{F}} v^k) \partial_k \mathbf{F} + v^k \nabla_{\partial_i \mathbf{F}} \partial_k \mathbf{F} = (\partial_i v^k) \partial_k \mathbf{F} + v^k \Gamma_{ik}^m \partial_m \mathbf{F}.$$

That is a vector, but it will be convenient to calculate relevant coordinates, motivating the definition

$$\nabla_i v^j := g^{jl} (\nabla_{\partial_i \mathbf{F}} V, \partial_l \mathbf{F}) = g^{jl} \left[g_{lk} \partial_i v^k + v^k \Gamma_{ik}^m g_{jm} \right] = \delta_k^j \partial_i v^k + \delta_m^j v^k \Gamma_{ik}^m = \partial_i v^j + v^k \Gamma_{ik}^j.$$

Since we think of H as adaptation of the Laplacian of $\mathbf{F}$, we need to define divergence on a hypersurface.

Then we naturally define the divergence of a vector field on M by

$$\operatorname{div}_M V = \nabla_i^M v^i.$$

Therefore the Laplace-Beltrami operator on M should be defined by

$$\Delta_M f := \operatorname{div}_M \nabla^M f = \nabla_i^M (\nabla^M f)^i = g^{ij} \nabla_i^M (\partial_j f) = g^{ij} (\partial_i \partial_j f - \Gamma_{ij}^k \partial_k H).$$

Although the derivation is not obvious from just what's here, for definiteness one can use the formula

$$\Delta_M f = \frac{1}{J} \partial_i (J g^{ij} \partial_j f),$$

which is calculable directly from the metric g.

Then finally we can write for the *mean curvature vector*

$$\mathbf{H} = \Delta_M \mathbf{F} = (\Delta_M \mathbf{F}^i)\partial_i \mathbf{F} + \Delta_M(\mathbf{F}^{N+1})\nu.$$

Furthermore take the *(scalar) mean curvature* H to be the signed magnitude of $\mathbf{H}$, that is

$$\mathbf{H} = -H\nu$$

so that a sphere has positive mean curvature.

Another approach to understanding the mean curvature is as the sum (rather, the non-normalized mean) of principal curvatures. Just as g is a function on M and its tangent spaces, define the second fundamental form A by the matrix of functions:

$$A_{ij} = \partial_i \nu \cdot \partial_j \mathbf{F} = -\partial_i \partial_j \mathbf{F} \cdot \nu,$$

which is a symmetric matrix. We can also calculate for the *Weingarten map* $W(V) := -V(\nu) := -\partial_V \nu$,

$$W^i_j = A^i_j = g^{ik} A_{kj},$$

another descriptor of curvature. Lower indices indicate components of covectors, which are real-valued functions that take vectors as input. We will not use covectors, but the following paragraph is to clarify the use of changing indices (see pp. 27-28 of [36] for a discussion of raising and lowering indices).

For example, $dp^i(V) = v^i$. This way the summation still works for a covector: $\omega = w_i dp^i$. Thus, that A^i_j has upper and lower indices means that it takes vectors and outputs vectors, like a linear matrix operator one would be familiar with. As seen in the equation for W, multiplication by G or G^{-1} can change the type of input an object takes. This is represented by raising or lowering and index. For example, $g_{ij}x^j = x_i$ gives the components of the covector $V^\flat = v_i dp^i$, which has the effect

$$V^\flat(W) = v_i dp^i(W) = v_i w^i = g_{ij} v^j w^i = v^j w^i \partial_i \mathbf{F} \cdot \partial_j \mathbf{F} = (v^j \partial_j \mathbf{F}) \cdot (w^i \partial_i \mathbf{F}) = V \cdot W.$$

This is known as lowering an index. Similarly, for a covector $\omega^\sharp = (\omega^\sharp)^i \partial_i \mathbf{F} = g^{ij} \omega_j \partial_j \mathbf{F}$. This is known as raising an index.

We use only for its eigenvalues κ_i. On a two-dimensional surface, κ_1 and κ_2 are the minimum and maximum curvatures at a point. So for a sphere they may both be 1 and at a saddle point they may be -1 and 1. We only use these as another way to understand mean curvature:

$$H = \sum_i \kappa_i = A_i^i = g^{ij} A_{ij} = g^{ij} \partial_i \nu \cdot \partial_j \mathbf{F} = \operatorname{div}_M \nu.$$

So in the previous examples of sphere and saddle point, we might have $H = 2$ for the sphere and $H = 0$ at the saddle point. Notice that $A_i^i = g^{ij} A_{ij}$ is the trace of A, $\operatorname{tr}_g A$. This is called a metric contraction. Another use of metric contraction is to define a norm on tensors such as A:

$$|A|_g^2 = \langle A, A \rangle = g^{ij} g^{kl} A_{ik} A_{jl}.$$

Note we then have

$$A^{ij} A_{ij} = g^{ik} g^{jl} A_{kl} A_{ij} = |A|_g^2.$$

B Evolution Equations

The purpose of this appendix is merely to give an outline of the derivation for the evolution of H, since it is used in the minimum principle in §2.3 and a flavor for the straightforwardness of the calculations, rather than a full account.

Since $H = \sum_{ij} g^{ij} A^{ij}$, we need evolution equations for g^{ij} and A_{ij}. Recall $g_{ij} = \partial_i \mathbf{F} \cdot \partial_j \mathbf{F}$. Thus, keeping in mind $\partial_i \mathbf{F}$ is tangent, and therefore $\partial_i \mathbf{F} \perp \nu$, we have

$$\begin{aligned}\partial_t g_{ij} &= \partial_t(\partial_i \mathbf{F} \cdot \partial_j \mathbf{F}) = \partial_i \partial_t \mathbf{F} \cdot \partial_j \mathbf{F} + \partial_i \mathbf{F} \cdot \partial_j \partial_t \mathbf{F} = \partial_i(-H\nu) \cdot \partial_j \mathbf{F} + \partial_i \mathbf{F} \cdot \partial_j(-H\nu) \\ &= -\cancel{(\partial_i H)\nu \cdot \partial_j \mathbf{F}} - H\partial_i \nu \cdot \partial_j \mathbf{F} - \cancel{(\partial_j H)\nu \cdot \partial_j \mathbf{F}} - H\partial_j \nu \cdot \partial_i \mathbf{F} = -2HA_{ij}.\end{aligned}$$

We also want the evolution for G^{-1}.

We can differentiate (A.1) with respect to t to find

$$0 = \partial_t(\delta_a^b) = \partial_t(g_{ab} g^{bc}) = \partial_t g_{ab} g^{bc} + g_{ab}(\partial_t g^{bc}) = -2Hg^{bc} A_{ab} + g_{ab}(\partial_t g^{bc}).$$

We want to isolate the derivative by moving the negative term to the other side and can "cancel" by multiplying by g^{ad} and summing over a. Let's do the two terms separately. First

$$g^{ad} g_{ab} \partial_t g^{bc} = g_b^d \partial_t g^{bc} = \partial_t g^{dc}.$$

Second

$$2Hg^{ad} g^{bc} A_{ab} = 2HA^{dc}.$$

Therefore

$$\partial_t g^{ij} = 2HA^{ij}.$$

Next, we want the evolution for A_{ij}. Note that since ν is always unit, any derivative is tangent to M, so $\partial_t \nu \perp \nu$ and $\partial_i \nu \perp \nu$. Thus

$$0 = \partial_i(\partial_j \nu \cdot \nu) = \partial_i \partial_j \nu \cdot \nu + \partial_i \nu \cdot \partial_j \nu$$

so $\partial_i \partial_j \nu \cdot \nu = -\partial_i \nu \cdot \partial_j$, and we can begin

$$\begin{aligned}
\partial_t A_{ij} &= -\partial_t(\partial_i \partial_j \mathbf{F} \cdot \nu) = -\partial_i \partial_j \partial_t \mathbf{F} \cdot \nu - \partial_i \partial_j \mathbf{F} \cdot \partial_t \nu = -\partial_i \partial_j(-H\nu) \cdot \nu - \partial_i \partial_j \mathbf{F} \cdot \partial_t \nu \\
&= (\partial_i \partial_j H)\nu \cdot \nu + \cancel{(\partial_i H)\partial_j \nu \cdot \nu} + \cancel{(\partial_j H)\partial_i \nu \cdot \nu} + H(\partial_i \partial_j \nu) \cdot \nu - \partial_i \partial_j \mathbf{F} \cdot \partial_t \nu \\
&= \partial_i \partial_j H - H \partial_i \nu \cdot \partial_j \nu - \partial_i \partial_j \mathbf{F} \cdot \partial_t \nu.
\end{aligned}$$

Clearly we need to make a few more calculations to wrap up the last line there. The second term is easier than the third. We have

$$\partial_i \nu \cdot \partial_j \nu = g^{km}(\partial_i \nu \cdot \partial_k \mathbf{F})\partial_m \mathbf{F} \cdot g^{ln}(\partial_j \nu \cdot \partial_l \mathbf{F})\partial_n \mathbf{F} = g^{km} g^{ln} g_{mn} A_{ik} A_{jl} = \delta^k_n g^{ln} A_{ik} A_{jl} = g^{lk} A_{ik} A_{jl}.$$

For the third term, let us first investigate $\partial_t \nu$. Since

$$\partial_t \cdot \partial_i \mathbf{F} = \cancel{\partial_t(\nu \cdot \partial_i \mathbf{F})} - \nu \cdot \partial_i \partial_t \mathbf{F} = -\nu \cdot \partial_i(-H\nu) = -(\partial_i H)\nu \cdot \nu - \cancel{H\nu \cdot \partial_i \nu} = -\partial_i H,$$

we can find that

$$\partial_t \nu = g^{ij}(\partial_t \nu \cdot \partial_i \mathbf{F})\partial_j \mathbf{F} = -g^{ij}(\partial_i H)\partial_j \mathbf{F} = -\nabla^M H.$$

Now for the third term:

$$\begin{aligned}
\partial_i \partial_j \mathbf{F} \cdot \partial_t \nu &= -(\partial_i \partial_j \mathbf{F})^\top \cdot \nabla^M H = -\Gamma^k_{ij} \partial_k \mathbf{F} \cdot \nabla^M H = -\Gamma^k_{ij} \partial_k \mathbf{F} \cdot g^{mn}(\partial_m H)\partial_n \mathbf{F} \\
&= -g^{mn} g_{nk} \Gamma^k_{ij} \partial_m H = -\partial^m_k \Gamma^k_{ij} \partial_m H = -\Gamma^k_{ij} \partial_k H.
\end{aligned}$$

Now we are ready to calculate the evolution of H.

$$\partial_t H = \partial_t(g^{ij}A_{ij}) = (\partial_t g^{ij})A_{ij} + g^{ij}(\partial_t A_{ij}) = 2HA^{ij}A_{ij} + g^{ij}(\partial_i\partial_j H - Hg^{kl}A_{ik}A_{jl} - \Gamma^k_{ij}\partial_k H)$$
$$= g^{ij}(\partial_i\partial_j H - \Gamma^k_{ij}\partial_k H) + 2HA^{ij}A_{ij} - Hg^{ij}g^{kl}A_{ik}A_{jl} = \Delta_M H + |A|^2 H.$$

Since in this work we assume $H \geq 0$ at time 0, $\partial_t H \geq \Delta_M H$, so it is clear a minimum principle applies and M remains mean-convex.

C Graph Identities

Notation

- If $v \in \mathbb{R}^{N+1}$, then $v^2 := v \otimes v$.
- $u = \dfrac{1}{\sqrt{1+|Df|^2}}$.
- $|[a_{i_1,\ldots,i_k}]_{i_1,\ldots,i_k}|^2 = \displaystyle\sum_{i_1,\ldots,i_k} (a_{i_1,\ldots,i_k})^2$
- e_i: unit vector in the x_i direction
- $\partial_i f$ or $\partial^i f$: i-th derivative of f.

  (Superscript is only for the sake of Einstein notation.)

Matrix Facts

Algebra

If $v \in \mathbb{R}^n$, then $(v^2)^n = |v|^{2(n-1)} v^2$, for $n \geq 1$.

Note: $v^2 v^2 = \left[\displaystyle\sum_k v_i v_k v_k v_j\right]_{ij} = |v|^2 v^2$.

For induction, assume the hypothesis. Now

$$(v^2)^{n+1} = (v^2)^n \cdot v^2 = |v|^{2(n-1)} v^2 \cdot v^2 = |v|^{2n} v^2$$

(symmetric matrices commute). //

- Determinant: By Sylvester's determinant identity,

$$\det(I + v^2) = 1 + |v|^2.$$

Inverses

- $$(I + v^2)^{-1} = I - \frac{1}{1 + |v|^2} v^2.$$

  This is inspired by a geometric series argument. Assume for the moment that $I + fv^2$ is invertible, $\sum_{k=0}^{\infty} (fv^2)^k$ converges, and that the inverse is equal to that sum. It also makes sense to assume that $|f||v|^2 < 1$. Now

  $$\begin{aligned}
  (I + fv^2)^{-1} &= \sum_{k=0}^{\infty} (-fv^2)^k = I + \sum_{k=1}^{\infty} (-f)^k (v^2)^k = I + \left[\sum_{k=1}^{\infty} (-f)^k |v|^{2k} \right] |v|^{-2} v^2 \\
  &= I + \left[-1 + \sum_{k=0}^{\infty} (-f|v|^2)^k \right] |v|^{-2} v^2 = I - \left[1 - \frac{1}{1 + f|v|^2} \right] |v|^{-2} v^2 \\
  &= I - \frac{f}{1 + f|v|^2} v^2
  \end{aligned}$$

  Since the expression for the inverse is easily verifiable by algebra, we really only need the end term to be defined, or $f \neq -|v|^2$. This is, of course, satisfied if f is positive.

On Graphs

Metrics

Now we intend to apply the above to the metric of a hypersurface that is a graph over the hyperplane $\widehat{R}$. That is, a parameterization of the form

$$\mathbf{F}(x) = (x, f(x)).$$

We have

$$\partial_i \mathbf{F} = (e_i, \partial_i f),$$

so

$$g_{ij} = \partial_i \mathbf{F} \cdot \partial_j \mathbf{F} = \delta_{ij} + (\partial_i f)(\partial_j f),$$

or

$$G = I + (Df)^2.$$

Inverse Metric

Use the inverse formula above where $f = 1$ and $v = Df$. Then,

$$G^{-1} = [I + (Df)^2]^{-1} = I - \frac{(Df)^2}{1 + |Df|^2} = I - u^2 (Df)^2,$$

or

$$g^{ij} = \delta^{ij} - \frac{\partial^i f \partial^j f}{1 + |Df|^2} = \delta^{ij} - u^2 \partial^i f \partial^j f.$$

Christoffel Symbols

$$\begin{aligned}
\Gamma^k_{ij} &= \frac{g^{kl}}{2}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \\
&= \frac{1}{2}\left(\delta^{kl} - \frac{\partial^k f \partial^l f}{1 + |Df|^2}\right)[(\partial_i \partial_j f)(\partial_l f) + \cancel{(\partial_j f)(\partial_i \partial_l f)} + (\partial_j \partial_i f)(\partial_l f) + \cancel{(\partial_i f)(\partial_j \partial_l f)} \\
&\qquad - \cancel{(\partial_l \partial_i f)(\partial_j f)} - \cancel{(\partial_i f)(\partial_l \partial_j f)}] \\
&= \left[\delta^{kl} - \frac{(\partial^k f)(\partial^l f)}{1 + |Df|^2}\right](\partial_i \partial_j f)(\partial_l f) \\
&= (\partial_i \partial_j f)\left(\partial^k f - \frac{|Df|^2 \partial^k f}{1 + |Df|^2}\right) \\
&= (\partial^k f)(\partial_i \partial_j f)\left(1 - \frac{|Df|^2}{1 + |Df|^2}\right) = u^2 (\partial^k f)(\partial_i \partial_j f)
\end{aligned}$$

Curvature

We find

$$\partial_i \partial_j \mathbf{F} = (0, \partial_i \partial_j f).$$

We know

$$\nu = \frac{(-Df,1)}{\sqrt{1+|Df|^2}} = u(-Df,1).$$

So for outward normal pointing upward:

$$A_{ij} = -\partial_i\partial_j\mathbf{F}\cdot\nu = -(0,\partial_i\partial_j f)\cdot\frac{(-Df,1)}{\sqrt{1+|Df|^2}} = -\frac{\partial_i\partial_j f}{\sqrt{1+|Df|^2}} = -u\partial_i\partial_j f.$$

(Reverse sign if graphing from the outside.)

Or

$$A = -\frac{D^2 f}{\sqrt{1+|Df|^2}} = -uD^2 f.$$

Norm

We first calculate

$$\delta^{ik}\delta^{jl}(\partial_i\partial_j f)(\partial_k\partial_l f) = \sum_{ij}(\partial_i\partial_j f)^2 = |D^2 f|^2,$$

$$\delta^{ik}(\partial^j f)(\partial^l f)(\partial_i\partial_j f)(\partial_k\partial_l f) = (\partial^j f)(\partial^l f)\sum_i(\partial_i\partial_j f)(\partial_i\partial_l f) = Df\cdot(D^2 f)^2\cdot Df.$$

$$(\partial^i f)(\partial^k f)(\partial^j f)(\partial^l f)(\partial_i\partial_j f)(\partial_k\partial_l f) = \left[Df\cdot(D^2 f)\cdot Df\right]^2.$$

Then

$$\begin{aligned}
|A|^2 &= g^{ik}g^{jl}A_{ij}A_{kl}\\
&= \left[\delta^{ik} - u(\partial^i f)(\partial^k f)\right]\left[\delta^{jl} - u(\partial^j f)(\partial^l f)\right][u\partial_i\partial_j f][u\partial_k\partial_l f]\\
&= u^2\left[\delta^{ik}\delta^{jl} - u\delta^{ik}(\partial^j f)(\partial^l f) - u\delta^{jl}(\partial^i f)(\partial^k f) + u^2(\partial^i f)(\partial^k f)(\partial^j f)(\partial^l f)\right](\partial_i\partial_j f)(\partial_k\partial_l f)\\
&= u^2\left[|D^2 f|^2 - 2u\left(Df\cdot(D^2 f)^2\cdot Df\right) + u^2\left(Df\cdot(D^2 f)\cdot Df\right)^2\right]
\end{aligned}$$

Higher Derivatives

$$
\begin{aligned}
(\nabla^k A)(\partial_{a_1}, \ldots, \partial_{a_{k+2}}) &= \left(\nabla(\nabla^{k-1}A)\right)(\partial_{a_1}, \ldots, \partial_{a_{k+2}}) \\
&= \nabla_{\partial_{a_{k+2}}}\left((\nabla^{k-1}A)(\partial_{a_1}, \ldots, \partial_{a_{k+1}})\right) \\
&\qquad - \sum_i (\nabla^{k-1}A)(\partial_{a_1}, \ldots, \nabla_{\partial_{a_{k+2}}}\partial_{a_i}, \ldots, \partial_{a_{k+1}}) \\
&= \partial_{a_{k+2}}(\nabla^{k-1}A_{a_1\cdots a_{k+1}}) - \sum_i \Gamma^l_{a_i a_{k+2}} \nabla^{k-1} A_{a_1 \cdots \underset{i-th}{\frown{l}} \cdots a_{k+1}} \\
&= \partial_{a_{k+2}}(\nabla^{k-1}A_{a_1\cdots a_{k+1}}) \\
&\qquad - u^2(\partial^l f)\sum_i (\partial_{a_{k+2}}\partial_{a_i} f)\nabla^{k-1} A_{a_1 \cdots \underset{i-th}{\frown{l}} \cdots a_{k+1}}
\end{aligned}
$$

In particular,

$$
\begin{aligned}
\nabla A_{ijk} &= \partial_k A_{ij} - \Gamma^l_{ik} A_{jl} - \Gamma^l_{jk} A_{il} \\
&= \partial_k A_{ij} - u^2(\partial^l f)\left[(\partial_k \partial_i f) A_{jl} + (\partial_k \partial_j f) A_{il}\right]
\end{aligned}
$$

D Weak and Strong Convergence

For a smooth, compact hypersurface Σ embedded in $\mathbb{R}^{N+1}$, define the measure associated to Σ by

$$\eta(A) = \mathcal{H}^N(\Sigma \cap A),$$

so that η is concentrated on only Σ and is fairly weighted across Σ. Of course Σ has finite $\mathcal{H}^N$-measure and the topology inherited from $\mathbb{R}^{N+1}$, so η is a Radon measure and integrals are therefore defined and finite for functions in $C_c^0(\mathbb{R}^{N+1})$. We say $\eta_n \to \eta$ if

$$\int_{\mathbb{R}^{N+1}} g \, d\eta_n \to \int_{\mathbb{R}^{N+1}} g \, d\eta$$

for every $g \in C_c^0(\mathbb{R}^{N+1})$.

Lemma. *Let Σ_n and Σ be embedded closed hypersurfaces in $\mathbb{R}^{N+1}$ and η_n and η be the Radon measures associated associated to Σ and Σ_n, respectively.*

If $\Sigma_n \to \Sigma$ as graphs, $\eta_n \to \eta$ in the sense of Radon measures.

Proof. Let $\varepsilon > 0$. We have that for sufficiently large n, Σ_n is a graph of f_n over Σ, so it is the image of $\varphi_n : \Sigma \to \Sigma_n$ defined by

$$\varphi_n(x) = x + f_n(x)\nu(x),$$

and that $\|f\|_{C^1} < \varepsilon$. Now write the Jacobian $J_n = |\det(\nabla\varphi_n)|$. We'll need to understand J_n to compare measures.

Since Σ is smooth and compact, $\partial_i \nu^j$ is bounded, so we have

$$\partial_i \varphi_n^j = \delta_i^j + (\partial_i f)\nu^j + f\partial_i\nu^j) = \delta_i^j + O(\varepsilon).$$

Then

$$\det(\nabla\varphi_n) = \sum_{\sigma\in S_N} \Pi_{i=1}^N \partial_i \varphi_{\sigma_i} = \Pi_{i=1}^N (1 + O(\varepsilon)) = 1 + O(\varepsilon),$$

where S_N is the set of permutations of $\{1, 2, \ldots, N\}$. Note that this means $|J_n| < 2$.

Now take $g \in C_c^0(\mathbb{R}^{N+1}, \mathbb{R})$. That means that since g is uniformly continuous and

$$|\varphi_n(x) - x| = |f_n(x)\nu(x)| < \varepsilon,$$

we know $|g(\varphi_n(x)) - x| = O(\varepsilon)$. Finally we have

$$\begin{aligned}
\left| \int_{\mathbb{R}^{N+1}} g \, d\eta_n - \int_{\mathbb{R}^{N+1}} g \, d\eta \right| &= \left| \int_{\Sigma_n} g \, d\eta_n - \int_{\Sigma} g \, d\eta \right| \\
&= \left| \int_{\Sigma} g(\varphi_n(x)) J_n(x) - g(x) \, d\mu \right| \\
&= \left| \int_{\Sigma} g(\gamma_n(x)) J_n(x) - g(x) J_n(x) + g(x) J_n(x) - g(x) \, d\mu \right| \\
&= \left| \int_{\Sigma} [g(\varphi_n(x)) - g(x)] J_n(x) + g(x)[J_n(x) - 1] \, d\mu \right| \\
&< 2 \int_{\Sigma} O(\varepsilon) \, d\mu + O(\varepsilon) \int_{\Sigma} g(x) \, d\mu \\
&= O(\varepsilon).
\end{aligned}$$

Since ε was arbitrary, we are done.

$\square$

Vita

Kevin Sonnanburg was born in Milford, Michigan to Keith and Janet Sonnanburg, who instilled an early interest in mathematics. He soon moved to scenic Mukilteo, Washington where he attended Kamiak High School. There his AP humanities teachers, Julie Russell and Shan Oglesby, taught education can be taken into one's own hands. Kevin went on to Whitworth University in Spokane where he participated in the Whitworth Choir all four years and was math major of the year. He was able to work with Nick Willis on projects in algebraic geometry and topology and Nate Moyer in number theory. That year he met Kevin Vixie who offered a research internship at Washington State University using geometry to improve visual simulations and medical imagery. After college he participated in the PCMI summer workshop in geometric image analysis. Thus developed his interest in geometric analysis. Luckily his advisor Alex Freire offered in sequence courses in Riemannian geometry, Ricci flow, mean curvature flow and related seminars. Kevin has an affinity for drawing concrete pictures, leading to this dissertation in mean curvature flow embedded in Euclidean space.